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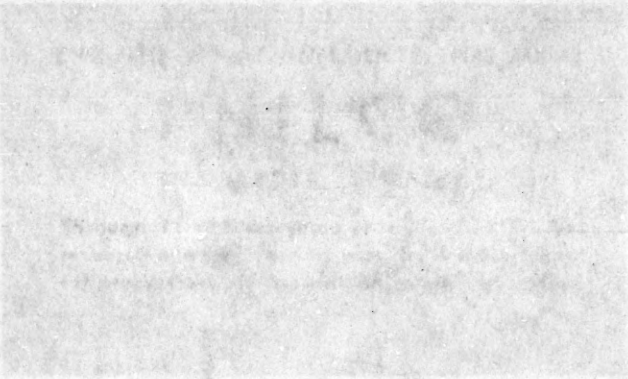

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TENSORIAL ANALYSIS AND EQUIVALENT CIRCUITS OF ELASTIC STRUCTURES.

BY

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SYNOPSIS.

Equivalent circuits are developed for one-, two-, and three-dimensional elastic structures that enable their solution by means of a Network Analyzer under steady small deformation and under forced or natural vibration. The elastic structure is assumed to consist of long, thin beams and rigid bodies having different elastic coefficients (tension and shear, bending and twist) along three perpendicular directions and interconnected into an arbitrary network either rigidly or by various constraints (pin joints, hinges, rollers, etc.) in any combination.

For the approximate numerical solution of structures Southwell's relaxation method is given, and for their exact solution a tensorial method is presented. Both methods employ matrices to systemize and speed up the calculations.

A companion paper presents numerical checks and Network Analyzer solutions of two-dimensional structures.

INTRODUCTION.

It has been known for over a hundred years that—assuming motion or rotation along one axis only—mass is analogous to inductance, and spring to capacitance, also that velocity is analogous to current and force to voltage. The idea of replacing a vibrating mechanical network of elastic bodies by an analogous electrical network has led to new inventions and the broadening of engineering understanding. The arrival of the A-C Network Analyzer has accelerated the practical use of these analogies, but all known applications still retain the one-dimensional character of the analogy.

However, the individual members of mechanical structures have in general not one degree of freedom but six. For instance, each elastic member of a framework may be elongated not only in the x-direction (or twisted not only around the x-axis) but may be elongated in the y- and z-directions as well and subjected simultaneously to rotations around the x, y, and z axes. *One of the purposes of this paper is to extend the electrical analogy to mechanical structures in which each member has six, instead of one, degrees of freedom,* retaining at the same time as much of the simplicity of the one-dimensional analogue as possible. The extended network reduces as a special case to known networks in the case of units with one degree of freedom. Another purpose of the

paper is to serve as an introduction to a further extension of the electrical analogy to general dynamical systems in which elastically and inelastically connected rigid bodies have relative rotation and translation with respect to each other.

Each elastic member in a frame structure is considered as a six-wire transmission line and a network of elastic members is considered as a similar network of interconnected six-wire transmission systems subjected to unbalanced loadings at various points along the line. Thereby, a large portion of the theory of electrical transmission lines developed by electrical engineers may be taken over by the mechanical engineer for the solution of elastic frames.

It must be emphasized that the mechanical transmission system is far more complicated than the electrical one. Each component mechanical beam is represented by a six-wire transmission line, while the electrical engineer deals chiefly with a three-wire line of a three-phase power system. (In a two-dimensional plane frame or grid frame, however, the mechanical system also consists of only three-wire transmission lines.) Also, each of the transmission lines and each of the loadings are in general of the unbalanced type, different in each of the six phases. Even the thought of using conventional scalar methods for the analysis of such a system is appalling.

The theory of complicated stationary structures in which each component itself is a complex structure has already been developed by the writer in a book entitled "Tensor Analysis of Networks" in which the scalar manipulations are replaced by matric manipulations and the scalar reasonings by tensorial reasonings. That method of analysis will be bodily taken over and employed here practically unchanged. The casual reader of this paper will only need to have some faint idea of what a matrix is and that matrices may be added and multiplied just as scalars can be. He should be able to follow the trend of thought intelligently. For the benefit of the more serious reader constant reference will be given throughout the paper to the above book as T. A. N., to save the repetition of what has already been published.

The analysis of a long, thin beam follows closely that of Southwell (reference 1) whose assumptions and nomenclature are used wherever possible.

The equivalent circuits developed enable the engineer to solve many of his steady-state and small-oscillation problems on already available A-C Network Analyzers, as the circuits require only inductive coils and capacitors. It is also felt that for an *approximate* solution by the relaxation method the physical picture offered by the use of circuits and the systematization introduced by the use of matrices should speed up the numerical work.

Of course, the *direct* analytical method shown here for the exact solution of a framework leads to just as prohibitively large number of

linear equations as any other exact method would. However, it is hoped that because of the systematic matrix form of the equations and the available reduction formulae to simplify them, and also because of the easily visualizable physical picture implied by the method of attack, the engineer well versed in the art of structural analysis will be in a better position to make simplifying assumptions and introduce other labor-saving processes. While one labor-saving procedure well known among electrical engineers, the concepts of "General Circuit Parameters" is introduced here, it seems to the writer that some method analogous to the "Method of Symmetrical Components" used universally for the study of electrical transmission systems, is inherent also in mechanical structures and is awaiting its discoverer.

There has been one attempt made by Vannevar Bush (reference 3) to establish equivalent circuits for elastic structures, but with the aid of ideal transformers (current transformers) in place of inductors and capacitors. He considered only special cases, in particular he assumed a two-dimensional frame with rigid joints having all its members at right angles and subjected to a shear and bending force but no elongation. The method of attack and the equivalent circuits presented here are based on entirely different principles.

PART I. SIMPLE BAR AS A TRANSMISSION LINE.

Reduction of a Structure to its Simplest Elements.

The purpose of the following study is to express the analysis of three-dimensional elastic structures in terms of the minimum number of concepts and mathematical symbols. That is accomplished by studying first in detail the simplest elements of the structure in their simplest manifestation. Then by giving each symbol increasingly wider interpretations with the aid of several "generalization postulates," the more complex analysis automatically unfolds to give the resultant physical model (equivalent circuit) or mathematical model (set of algebraic equations) of the structure.

An elastic structure is assumed to consist of masses concentrated in points and interconnected by elastic links consisting of long thin bars, to be called "beams." The simplest but still most general elements of the structure are found in three steps:

(1) All interconnections between the beams are removed, leaving a number of isolated but spatially oriented long, thin beams and a number of rigid bodies.

(2) Each beam and rigid body is rotated to lie along the x-axis with their three principal axes oriented along the x–y–z axes.

(3) Since at each end of a beam actually six forces act, for the preliminary study it will be assumed that only *one* force acts at each end of the beam and only one force acts on the point-mass.

Hence the first step in the analysis will be to establish the equations of two types of units, namely:

(1) a mass-point acted on by a force at its center of gravity;

(2) an elastic beam lying along the x-axis and acted on by a tensile force at both ends.

The electrical analogue of the first unit is a coil, Fig. 1a, that of the second unit a transmission line, Fig. 1b.

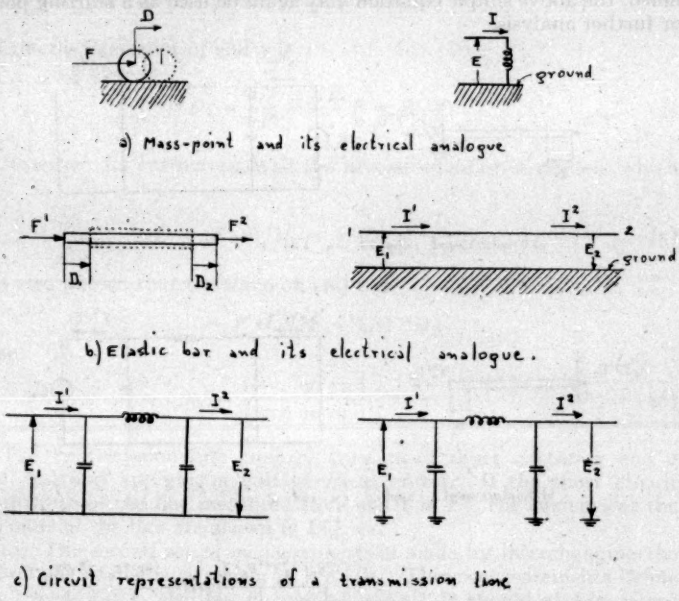


FIG. 1. Electrical analogues of one-dimensional mechanical systems.

Equation of a Coil.

The currents I will be assumed to represent forces F and the voltages E will be assumed to represent displacements D. The equation of a mass point and of an electrical coil will be assumed to be

$$F = M\frac{d^2D}{dt^2} = Mp^2D \qquad I = M\frac{d^2E}{dt^2} = Mp^2E \qquad (1)$$

$$F = YD \qquad I = YE \qquad (2)$$

where $p = d/dt$.

Since the tensile force F on both ends of a beam is the same, it appears that the electrical analogue of such a case is also an ordinary coil. However, in determining the necessary constants, the ends of a beam cannot be fixed or freed at will for each of the various types of forces and couples as an electrical network can. This limitation of possible measurements on a beam requires the introduction of additional concepts. Once the constants of the beam have been determined, the above simple equation may again be used as a starting point for further analysis.

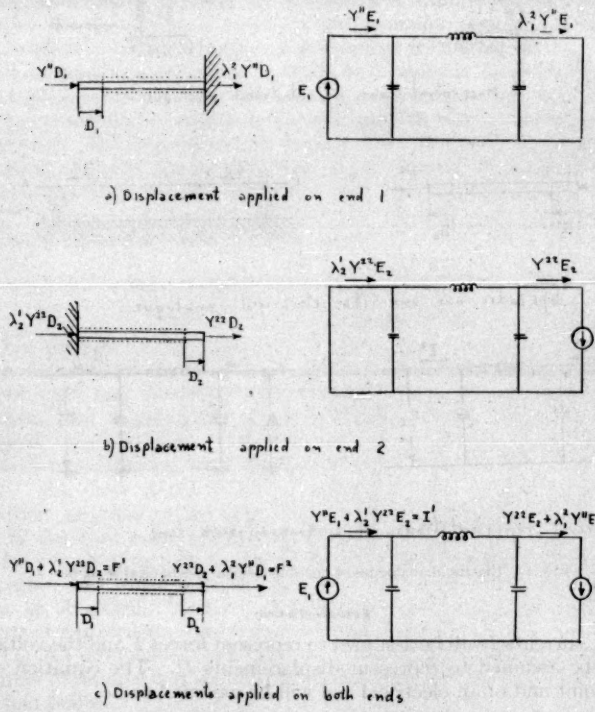


FIG. 2. Scalar equations of an one-dimensional bar.

Equations of a Transmission Line.

The sending and receiving ends of a transmission line, Fig. 1b, are denoted by 1 and 2. The currents will correspond to the forces and

the voltages to the displacements of the beam. The equations of a transmission system (and those of the beam) may be established by two sets of measurements.

(1) At first let end 2 of the beam be fixed (Fig. 2a) and a displacement D_1 be applied at end 1. The stored elastic energy is

$$U = \frac{1}{2}\frac{AE}{L}F^2 \tag{3}$$

and the displacement of end 1 is

$$D_1 = \frac{\partial U}{\partial F} = \frac{AE}{L}F = Z'_{11}F. \tag{4}$$

However, for further analysis the inverse equation is needed, which is

$$F = (Z'_{11})^{-1}D_1 = Y^{11}D_1 = \frac{L}{AE}D_1. \tag{5}$$

It is also known that the force on end 2 is

$$-F = \lambda_1^2 F = \lambda_1^2 Y^{11} D_1,$$

where

$$\lambda_1^2 = \frac{\text{force on end 2}}{\text{force on end 1}} = -1. \tag{6}$$

The transmission line analogy consists of short circuiting end 2 (Fig. 2a) and applying a voltage E_1 at end 1. If the short circuit admittance of the line measured from end 1 is Y^{11}, the currents at the two ends of the line are shown in Fig. 2a.

(2) The second set of measurements is made by interchanging the role of the two ends as shown in Fig. 2b. These measurements define Z_{22}, $Y^{22} = (Z_{22})^{-1}$, also D_2, E_2 and $\lambda_2^1 = -1$. It should also be noted that $\lambda_1^2\lambda_2^1 = 1$.

If now both ends of the beam are subjected to displacements simultaneously (or voltages are impressed on both ends of the line), the *resultant forces* (currents) on each end of the beam (line) are the sum of the previous cases as shown on Fig. 2c. That is, for the beam

$$\begin{aligned} F^1 &= Y^{11}D_1 + \lambda_2^1 Y^{22} D_2, \\ F^2 &= \lambda_1^2 Y^{11} D_1 + Y^{22} D_2. \end{aligned} \tag{7}$$

If now mutual admittances are defined as

$$Y^{12} = \lambda_2^1 Y^{22} \quad \text{and} \quad Y^{21} = \lambda_1^2 Y^{11}, \tag{8}$$

the equations of the beam become

$$F^1 = Y^{11}D_1 + Y^{12}D_2, \qquad (9)$$
$$F^2 = Y^{21}D_1 + Y^{22}D_2.$$

Similarly for the transmission line

$$I^1 = Y^{11}E_1 + Y^{12}E_2, \qquad (10)$$
$$I^2 = Y^{21}E_1 + Y^{22}E_2.$$

It is interesting that these equations cannot be solved for D or E as the determinant is zero.

Hence for the analysis of the simplest case of a beam, thirteen scalar symbols have hitherto been introduced:

1. The stored elastic energy U used as a starting point.
2. Z'_{11} and Z'_{22} defined by Eq. (4).
3. Y^{11} and Y^{22} defined by Eq. (5).
4. λ_1^2 and λ_2^1 defined by Eq. (6).
5. Y^{12} and Y^{21} defined by Eq. (8).
6. D_1, D_2; F^1, F^2 (or E_1, E_2; I^1, I^2).
7. The beam or line equations (9) or (10).

PART II. THE UN-ORIENTED BEAM.

The "First Generalization Postulate."

In general, each end of the beam has six degrees of freedom instead of one, namely,

1. three linear displacements D_x, D_y, D_z;
2. three angular displacements D_p, D_q, D_r.

Correspondingly there are six forces (three linear forces and three torques), a large number of design constants, and complicated relations between all of them. To simplify the involved analysis it will be postulated that (T. A. N., p. 47):

"All reasoning and equations developed for the beam with one degree of freedom will also be valid for a beam with six degrees of freedom provided each symbol in the reasoning and in the equations is replaced by an appropriate matrix."

In order to do that let the reasonings of the previous sections be repeated for the general beam, still, however, lying along the x-axis with its principal axes still oriented along x, y, z.

As the first set of measurements, let end 2 of the beam be fixed and end 1 be subjected to linear and angular displacements. The stored elastic energy is (reference 1, page 72)

$$U = \frac{1}{2}\int_0^L \left[\frac{(F^x)^2}{EA} + \frac{(F^q + rF^z)^2}{B_1} + \frac{(F^r - rF^y)^2}{B_2} + \frac{(F^p)^2}{C}\right] dr, \qquad (11)$$

where B_1 = flexural rigidity about axis y,
B_2 = flexural rigidity about axis z,
C = torsional rigidity.

Then by differentiating U with respect to each of the forces, six equations may be written as

$$
\begin{aligned}
D_x &= (L/EA)F^x \\
D_y &= (L^3/3B_2)F^y - (L^2/2B_2)F^r \\
D_z &= (L^3/3B_1)F^z + (L^2/2B_1)F^q \\
D_p &= (L/C)F^p \\
D_q &= (L^2/2B_1)F^z + (L/B_1)F^q \\
D_r &= -(L^2/2B_2)F^y + (L/B_2)F^r
\end{aligned}
$$

or in terms of matrices (T. A. N., Chapter I)

$$\mathbf{D}_1 = \mathbf{Z'}_{11}\mathbf{F}^1 \quad \text{or} \quad [D_1] = [Z'_{11}][F^1] \tag{12}$$

$$\mathbf{D}_1 = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & \mathbf{p}_1 & \mathbf{q}_1 & \mathbf{r}_1 \\ \hline D_{x1} & D_{y1} & D_{z1} & D_{p1} & D_{q1} & D_{r1} \\ \hline \end{array}$$

$$\mathbf{F}^1 = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & \mathbf{p}_1 & \mathbf{q}_1 & \mathbf{r}_1 \\ \hline F^{x1} & F^{y1} & F^{z1} & F^{p1} & F^{q1} & F^{r1} \\ \hline \end{array}$$

$$\mathbf{Z'}_{11} = \begin{array}{c|c|c|c|c|c|c|} & \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & \mathbf{p}_1 & \mathbf{q}_1 & \mathbf{r}_1 \\ \hline \mathbf{x}_1 & L/EA & & & & & \\ \hline \mathbf{y}_1 & & L^3/3B_2 & & & & -L^2/2B_2 \\ \hline \mathbf{z}_1 & & & L^3/3B_1 & & L^2/2B_1 & \\ \hline \mathbf{p}_1 & & & & L/C & & \\ \hline \mathbf{q}_1 & & & L^2/2B_1 & & L/B_1 & \\ \hline \mathbf{r}_1 & & -L^2/2B_2 & & & & L/B^2 \\ \hline \end{array}$$

In the electrical analogy (Fig. 3a) there are now six transmission lines and one ground. End 2 of all lines is short circuited and voltages $E_{x1}, E_{y1}, \cdots$, are impressed on end 1 in succession while *all the other end 1 terminals are kept open circuited* (corresponding to the free end of the beam). The matric equation of the transmission line is

$$\mathbf{E}_1 = \mathbf{Z'}_{11}\mathbf{I}^1 \tag{13}$$

$$\mathbf{E}_1 = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & \mathbf{p}_1 & \mathbf{q}_1 & \mathbf{r}_1 \\ \hline E_{x1} & E_{y1} & E_{z1} & E_{p1} & E_{q1} & E_{r1} \\ \hline \end{array}$$

$$\mathbf{I}^1 = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & \mathbf{p}_1 & \mathbf{q}_1 & \mathbf{r}_1 \\ \hline I^{x1} & I^{y1} & I^{z1} & I^{p1} & I^{q1} & I^{r1} \\ \hline \end{array}$$

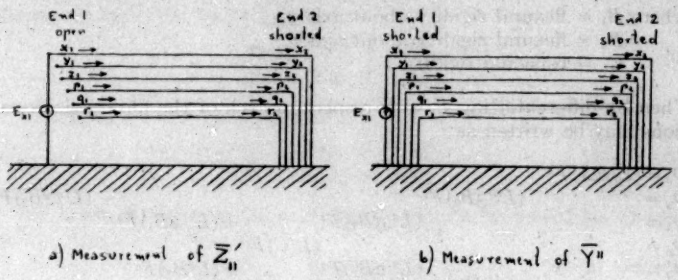


FIG. 3. Electrical analogue of a long, thin beam.

For further analysis the inverse equation is needed representing the reaction of a beam, in which both ends are rigidly connected to neighboring beams. The inverse equation is

$$\mathbf{F}^{\mathrm{I}} = (\mathbf{Z}'_{11})^{-1}\mathbf{D}_1 = \mathbf{Y}^{\mathrm{II}}\mathbf{D}_1 \qquad \mathbf{I}^{\mathrm{I}} = (\mathbf{Z}_{11})^{-1}\mathbf{E}_1 = \mathbf{Y}^{\mathrm{II}}\mathbf{E}_1$$

$$\mathbf{Y}^{\mathrm{II}} = \begin{array}{c|cccccc} & x_1 & y_1 & z_1 & p_1 & q_1 & r_1 \\ \hline x_1 & EA/L & & & & & \\ y_1 & & 12\,B_2/L^3 & & & & 6\,B_2/L^2 \\ z_1 & & & 12\,B_1/L^3 & & -6\,B_1/L^2 & \\ p_1 & & & & C/L & & \\ q_1 & & & -6\,B_1/L^2 & & 4\,B_1/L & \\ r_1 & & 6\,B_2/L^2 & & & & 4\,B_2/L \end{array} \qquad (14)$$

In the electrical analogy, $\mathbf{Y}^{\mathrm{II}}$ represents the self and mutual admittances of end 1 terminals measured while *all the other end 1 and end 2 terminals are short circuited* (Fig. 3b).

It is also known that the relations between the forces at the two ends of the beam are

$$\begin{aligned} F^{x2} &= -F^{x1} \\ F^{y2} &= -F^{y1} \\ F^{z2} &= -F^{z1} \\ F^{p2} &= -F^{p1} \\ F^{q2} &= -LF^{z1} - F^{q1} \\ F^{r2} &= LF^{y1} - F^{r1} \end{aligned}$$

In terms of matrices these may be written as

$$\mathbf{F}^2 = \lambda_1^2\mathbf{F}^1 = \lambda_1^2\mathbf{Y}^{11}\mathbf{D}_1 \tag{15}$$

$\mathbf{F}^2 =$

$\mathbf{x}_2$	$\mathbf{y}_2$	$\mathbf{z}_2$	$\mathbf{p}_2$	$\mathbf{q}_2$	$\mathbf{r}_2$
F^{x2}	F^{y2}	F^{z2}	F^{p2}	F^{q2}	F^{r2}

$\lambda_1^2 =$

	$\mathbf{x}_1$	$\mathbf{y}_1$	$\mathbf{z}_1$	$\mathbf{p}_1$	$\mathbf{q}_1$	$\mathbf{r}_1$
$\mathbf{x}_2$	-1					
$\mathbf{y}_2$		-1				
$\mathbf{z}_2$			-1			
$\mathbf{p}_2$				-1		
$\mathbf{q}_2$			$-L$		-1	
$\mathbf{r}_2$		L				-1

It should be noted that the horizontal and vertical unit vectors of λ_1^2 belong to different ends of the beam.

Compound Networks.

In reasoning further, it would be very awkward to show on the diagrams for both the beam and the transmission line six different forces (voltages) and displacements (currents). In analogy to the matric representation of an array of numbers by one bold face letter, a similar representation of diagrams will be used in which one arrow will represent six currents, one mesh will represent six meshes, etc. (T. A. N., p. 482). Such diagrams will be drawn with heavy lines (Fig. 4b). With the aid of such "compound networks," Fig. 5a, showing the equivalent circuit of a general beam with its end 2 fixed, looks the same as Fig. 2a showing the equivalent circuit of a beam with one degree of freedom. *A compound network must be so designed, as to give the correct relations between the various matrices,* just as the conventional network gives the correct relations between the scalars.

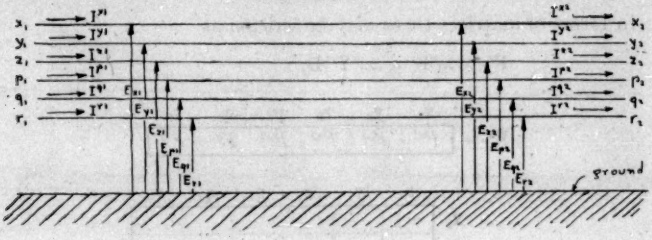
For the second set of measurements, end 1 of the beam is fixed and displacements are applied on end 2. In the elastic energy U, Eq. (11), the parameter r is now represented by $-r$, giving $\mathbf{D}_2 = \mathbf{Z'}_{22}\mathbf{F}^2$ where

$\mathbf{Z'}_{22} =$

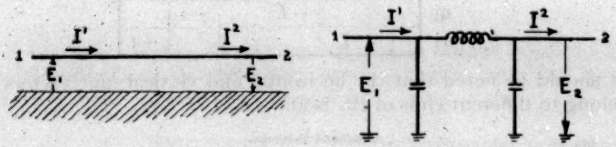
	$\mathbf{x}_2$	$\mathbf{y}_2$	$\mathbf{z}_2$	$\mathbf{p}_2$	$\mathbf{q}_2$	$\mathbf{r}_2$
$\mathbf{x}_2$	L/EA					
$\mathbf{y}_2$		$L^3/3B_2$				$L^2/2B_2$
$\mathbf{z}_2$			$L^3/3B_1$		$-L^2/2B_1$	
$\mathbf{p}_2$				L/C		
$\mathbf{q}_2$			$-L^2/2B_1$		L/B_1	
$\mathbf{r}_2$		$L^2/2B_2$				L/B_2

(16)

The inverse equation is $\mathbf{F}^2 = (\mathbf{Z'}_{22})^{-1}\mathbf{D}_2 = \mathbf{Y}^{22}\mathbf{D}_2$.



a) six-wire transmission line



b) "Compound" transmission line and its compound-circuit representation

FIG. 4. Representation of a complicated network by a simple "compound" network.

$$\mathbf{Y}^{22} = \begin{array}{c|cccccc} & x_2 & y_2 & z_2 & p_2 & q_2 & r_2 \\ \hline x_2 & EA/L & & & & & \\ y_2 & & 12\,B_2/L^3 & & & & -6\,B_2/L^2 \\ z_2 & & & 12\,B_1/L^3 & & 6\,B_1/L^2 & \\ p_2 & & & & C/L & & \\ q_2 & & & 6\,B_1/L^2 & & 4\,B_1/L & \\ r_2 & & -6\,B_2/L^2 & & & & 4\,B_2/L \end{array} \tag{17}$$

The forces on end 1 are $\lambda_2{}^1\mathbf{F}^2 = \lambda_2{}^1\mathbf{Y}^{22}\mathbf{D}_2$ where $\lambda_2{}^1 = (\lambda_1{}^2)^{-1}$ and is found from $\lambda_1{}^2$ by changing the sign of L.

$$\lambda_1{}^2 = \begin{array}{c|cccccc} & x_2 & y_2 & z_2 & p_2 & q_2 & r_2 \\ \hline x_1 & -1 & & & & & \\ y_1 & & -1 & & & & \\ z_1 & & & -1 & & & \\ p_1 & & & & -1 & & \\ q_1 & & & L & & -1 & \\ r_1 & & -L & & & & -1 \end{array} \tag{18}$$

The corresponding equivalent circuit is shown in Fig. 5b.

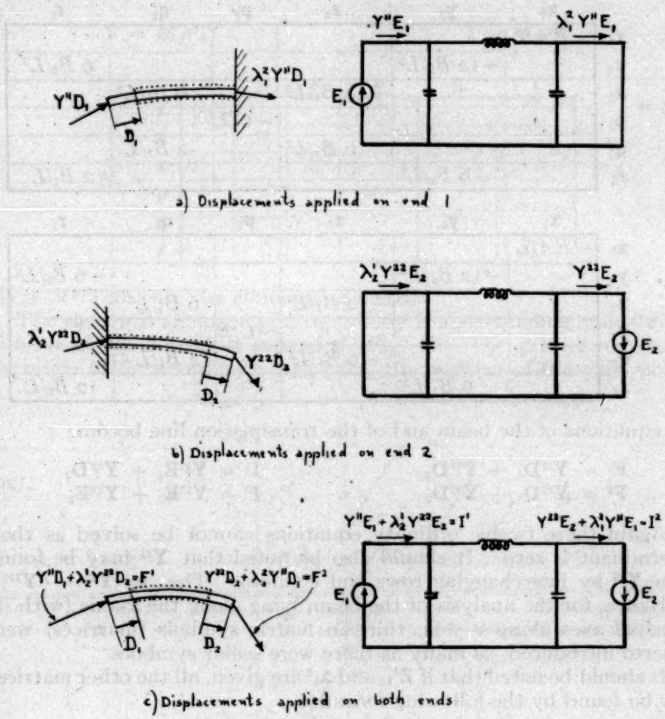


FIG. 5. Matrix equations of a long, thin beam.

Matric Equations of the Unoriented Beam.

When both ends of the beam are subjected to displacements simultaneously (or voltages are impressed on both ends of the line), the *resultant forces* (currents on each end of the beam (line)) are the sum of the previous cases, Fig. 5c. That is, for the beam

$$\mathbf{F}^1 = \mathbf{Y}^{11}\mathbf{D}_1 + \lambda_2{}^1\mathbf{Y}^{22}\mathbf{D}_2,$$
$$\mathbf{F}^2 = \lambda_1{}^2\mathbf{Y}^{11}\mathbf{D}_1 + \mathbf{Y}^{22}\mathbf{D}_2.$$

If now mutual admittance matrices are defined as

$$\mathbf{Y}^{12} = \lambda_2{}^1\mathbf{Y}^{22} \quad \text{and} \quad \mathbf{Y}^{21} = \lambda_1{}^2\mathbf{Y}^{11} \tag{19}$$

$$\mathbf{Y}^{12} = \begin{array}{c|cccccc} & \mathbf{x}_2 & \mathbf{y}_2 & \mathbf{z}_2 & \mathbf{p}_2 & \mathbf{q}_2 & \mathbf{r}_2 \\ \hline \mathbf{x}_1 & -EA/L & & & & & \\ \mathbf{y}_1 & & -12\,B_2/L^3 & & & & 6\,B_2/L^2 \\ \mathbf{z}_1 & & & -12\,B_1/L^3 & & -6\,B_1/L^2 & \\ \mathbf{p}_1 & & & & -C/L & & \\ \mathbf{q}_1 & & & 6\,B_1/L^2 & & 2\,B_1/L & \\ \mathbf{r}_1 & & -6\,B_2/L^2 & & & & 2\,B_2/L \end{array}$$

$$\mathbf{Y}^{21} = \begin{array}{c|cccccc} & \mathbf{x}_1 & \mathbf{y}_1 & \mathbf{z}_1 & \mathbf{p}_1 & \mathbf{q}_1 & \mathbf{r}_1 \\ \hline \mathbf{x}_2 & -EA/L & & & & & \\ \mathbf{y}_2 & & -12\,B_2/L^3 & & & & -6\,B_2/L^2 \\ \mathbf{z}_2 & & & -12\,B_1/L^3 & & 6\,B_1/L^2 & \\ \mathbf{p}_2 & & & & -C/L & & \\ \mathbf{q}_2 & & & -6\,B_1/L^2 & & 2\,B_1/L & \\ \mathbf{r}_2 & & 6\,B_2/L^2 & & & & 2\,B_2/L \end{array}$$

the equations of the beam and of the transmission line become

$$\begin{aligned} \mathbf{F}^1 &= \mathbf{Y}^{11}\mathbf{D}_1 + \mathbf{Y}^{12}\mathbf{D}_2 \\ \mathbf{F}^2 &= \mathbf{Y}^{21}\mathbf{D}_1 + \mathbf{Y}^{22}\mathbf{D}_2 \end{aligned} \qquad \begin{aligned} \mathbf{I}^1 &= \mathbf{Y}^{11}\mathbf{E}_1 + \mathbf{Y}^{12}\mathbf{D}_2 \\ \mathbf{I}^2 &= \mathbf{Y}^{21}\mathbf{E}_1 + \mathbf{Y}^{22}\mathbf{E}_2 \end{aligned} \tag{20}$$

Again these twelve ordinary equations cannot be solved as their determinant is zero. It should also be noted that $\mathbf{Y}^{21}$ may be found from $\mathbf{Y}^{12}$ by interchanging rows and columns. That is, $\mathbf{Y}^{21} = (\mathbf{Y}^{12})_t$.

Hence, for the analysis of the beam lying along the x-axis (with its principal axes along x–y–z), thirteen matric symbols (matrices) were hitherto introduced, as many as there were scalar symbols.

It should be noted that if $\mathbf{Z'}_{11}$ and λ_1^2 are given, all the other matrices may be found by the following formulae

$$\begin{aligned} \mathbf{Y}^{11} &= (\mathbf{Z'}_{11})^{-1} \\ \lambda_1^2 &= (\lambda_1^2)^{-1} \end{aligned} \qquad \begin{aligned} \mathbf{Y}^{21} &= \lambda_1^2\mathbf{Y}^{11} \\ \mathbf{Y}^{12} &= (\mathbf{Y}^{21})_t \end{aligned} \qquad \begin{aligned} \mathbf{Y}^{22} &= (\lambda_1^2)_t\mathbf{Y}^{11}\lambda_1^2 \\ \mathbf{Z'}_{22} &= (\lambda_2^1)_t\mathbf{Z'}_{11}\lambda_2^1 \end{aligned} \tag{21}$$

Also $\mathbf{Y}^{11} = (\mathbf{Y}^{11})_t$ and $\mathbf{Y}^{22} = (\mathbf{Y}^{22})_t$.

Matric Equations of a Rigid Body.

In a structure the masses of the beams often may be assumed to be concentrated at the junctions. When it is necessary to consider the mass of a beam concentrated at its center of gravity, then the beam is subdivided into two component elastic beams with a mass at their junction. Often a beam may be assumed to consist of several elastic beams in series with masses at their junctions.

For a rigid body with six degrees of freedom, by the first generalization postulate the force equation may be written in analogue to Eq. (1)

as

$$\mathbf{F} = \mathbf{M}p^2\mathbf{D} \qquad \mathbf{I} = \mathbf{M}p^2\mathbf{E} \qquad (22)$$

$$\mathbf{M} = \begin{array}{c|cccccc} & x & y & z & p & q & r \\ \hline x & M & & & & & \\ y & & M & & & & \\ z & & & M & & & \\ p & & & & M^{pp} & & \\ q & & & & & M^{qq} & \\ r & & & & & & M^{rr} \end{array}$$

where M^{pp}, etc., are the moments of inertia.

The electrical analogue of a rigid body is a six-winding transformer with six of the terminals grounded (Fig. 6). The ground represents the origin of the coordinate axes fixing the position of the mass point.

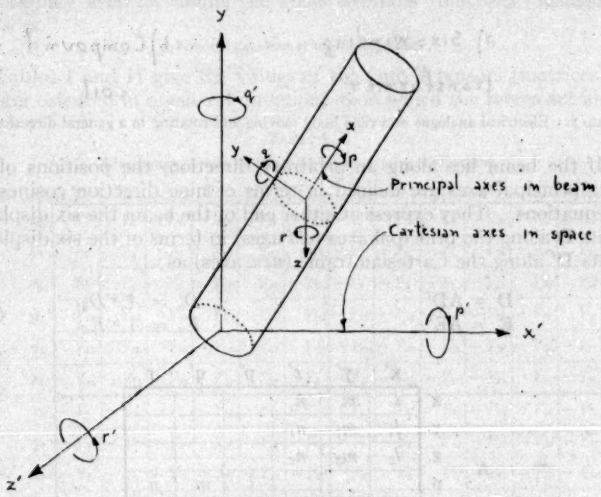


FIG. 6. Reference frames.

PART III. THE ORIENTED BEAM.

The "Second Generalization Postulate."

Instead of considering the beam lying along the x-axis of a Cartesian system, it may be said that the beam lies in an arbitrary direction in space, but the forces acting on it have hitherto been expressed not along

the axes of a Cartesian frame x', y', z' (Fig. 7) but along the three principal axes of the beam, x, y, z.

In order to express the equations of a beam in terms of forces acting not along its principal axes but along any arbitrary direction, it will be postulated that (T. A. N., p. 64):

"All reasoning and equations developed for the beam acted on by forces along its principal axes will also be valid for a beam acted on by forces along any arbitrary direction, provided each matrix in the reasoning and in the equations is replaced by an appropriate tensor."

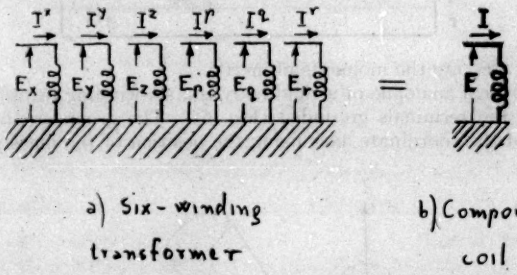


FIG. 7. Electrical analogue of a rigid body moving and rotating in a general direction.

If the beam lies along an arbitrary direction, the positions of its three principal axes are defined in terms of nine direction cosines by six equations. They express at either end of the beam the six displacements **D** along the principal axes (*old* axes) in terms of the six displacements **D′** along the Cartesian frame (*new* axes) as

$$\begin{aligned} \mathbf{D} &= \mathbf{A}\mathbf{D}' \\ \mathbf{E} &= \mathbf{A}\mathbf{E}' \end{aligned} \qquad \Big| \qquad \begin{aligned} D_\alpha &= A_\alpha{}^{\alpha'} D_{\alpha'} \\ E_\alpha &= A_\alpha{}^{\alpha'} E_{\alpha'} \end{aligned} \tag{23}$$

$$\mathbf{A} = \begin{array}{c|cccccc} & \mathbf{x}' & \mathbf{y}' & \mathbf{z}' & \mathbf{p}' & \mathbf{q}' & \mathbf{r}' \\ \hline \mathbf{x} & l & m & n & & & \\ \mathbf{y} & l_1 & m_1 & n_1 & & & \\ \mathbf{z} & l_2 & m_2 & n_2 & & & \\ \mathbf{p} & & & & l & m & n \\ \mathbf{q} & & & & l_1 & m_1 & n_1 \\ \mathbf{r} & & & & l_2 & m_2 & n_2 \end{array}$$

For either end of the beam the same **A** will be used to avoid the use of too many subscripts. (The two **A**-s, $\mathbf{A}_1{}^{1'}$ and $\mathbf{A}_2{}^{2'}$, differ in the subscripts of their unit vectors.) All quantities, when expressed along the new axes, will be primed.

The Laws of Transformation.

Since by a change of reference frame (with the aid of **A**), the stored elastic energy U does not change, the following relation exists

$$\begin{aligned} U &= \mathbf{DF} = \mathbf{D'F'} \\ U &= \mathbf{EI} = \mathbf{E'I'} \end{aligned} \qquad \begin{aligned} U &= D_\alpha F^\alpha = D_{\alpha'} F^{\alpha'} \\ U &= E_\alpha I^\alpha = E_{\alpha'} I^{\alpha'} \end{aligned} \tag{24}$$

From the last two relations follow (T. A. N., pp. 98–104) the laws of transformation

$$\begin{aligned} \mathbf{F'} &= \mathbf{A}_t\mathbf{F} \\ \mathbf{I'} &= \mathbf{A}_t\mathbf{I} \\ \mathbf{Y'} &= \mathbf{A}_t\mathbf{YA} \\ \boldsymbol{\lambda}' &= \mathbf{A}_t\boldsymbol{\lambda}\mathbf{A}^{-1} \\ \mathbf{M'} &= \mathbf{A}_t\mathbf{MA} \end{aligned} \qquad \begin{aligned} F^{\alpha'} &= A_\alpha{}^{\alpha'} F^\alpha \\ I^{\alpha'} &= A_\alpha{}^{\alpha'} I^\alpha \\ Y^{\alpha'\beta'} &= A_\alpha{}^{\alpha'} Y^{\alpha\beta} A_\beta{}^{\beta'} \\ \lambda_{\beta'}{}^{\alpha'} &= A_\alpha{}^{\alpha'} \lambda_\beta{}^\alpha A_{\beta'}{}^\beta \\ M^{\alpha'\beta'} &= A_\alpha{}^{\alpha'} M^{\alpha\beta} A_\beta{}^{\beta'} \end{aligned} \tag{25}$$

When a matrix is endowed with one of the above laws of transformation, it is called a "tensor." The reader not acquainted with tensors may keep on calling the above symbols "matrices" instead of "tensors."

Tensorial Equations of an Oriented Beam.

Tables I and II give the values of the four **Y** tensors (matrices) of a beam oriented in a general direction, upon which the forces act along

TABLE I.

Admittance Matrix of Beam Lying in Arbitrary Direction.

$\mathbf{Y} =$

	x_1	y_1	z_1	p_1	q_1	r_1	x_2	y_2	z_2	p_2	q_2	r_2
x_1	Y_1	Y_7	Y_8	Y_{19}	Y_{13}	Y_{14}	$-Y_1$	$-Y_7$	$-Y_8$	Y_{19}	Y_{13}	Y_{14}
y_1	Y_7	Y_2	Y_9	Y_{16}	Y_{20}	Y_{15}	$-Y_7$	$-Y_2$	$-Y_9$	Y_{16}	Y_{20}	Y_{15}
z_1	Y_8	Y_9	Y_3	Y_{17}	Y_{18}	Y_{21}	$-Y_8$	$-Y_9$	$-Y_3$	Y_{17}	Y_{18}	Y_{21}
p_1	Y_{19}	Y_{16}	Y_{17}	Y_4	Y_{10}	Y_{11}	$-Y_{19}$	$-Y_{16}$	$-Y_{17}$	$-Y_{22}$	$-Y_{25}$	$-Y_{26}$
q_1	Y_{13}	Y_{20}	Y_{18}	Y_{10}	Y_5	Y_{12}	$-Y_{13}$	$-Y_{20}$	$-Y_{18}$	$-Y_{25}$	$-Y_{23}$	$-Y_{27}$
r_1	Y_{14}	Y_{15}	Y_{21}	Y_{11}	Y_{12}	Y_6	$-Y_{14}$	$-Y_{15}$	$-Y_{21}$	$-Y_{26}$	$-Y_{27}$	$-Y_{24}$
x_2	$-Y_1$	$-Y_7$	$-Y_8$	$-Y_{19}$	$-Y_{13}$	$-Y_{14}$	Y_1	Y_7	Y_8	$-Y_{19}$	$-Y_{13}$	$-Y_{14}$
y_2	$-Y_7$	$-Y_2$	$-Y_9$	$-Y_{16}$	$-Y_{20}$	$-Y_{15}$	Y_7	Y_2	Y_9	$-Y_{16}$	$-Y_{20}$	$-Y_{15}$
z_2	$-Y_8$	$-Y_9$	$-Y_3$	$-Y_{17}$	$-Y_{18}$	$-Y_{21}$	Y_8	Y_9	Y_3	$-Y_{17}$	$-Y_{18}$	$-Y_{21}$
p_2	Y_{19}	Y_{16}	Y_{17}	$-Y_{22}$	$-Y_{25}$	$-Y_{26}$	$-Y_{19}$	$-Y_{16}$	$-Y_{17}$	Y_4	Y_{10}	Y_{11}
q_2	Y_{13}	Y_{20}	Y_{18}	$-Y_{25}$	$-Y_{23}$	$-Y_{27}$	$-Y_{13}$	$-Y_{20}$	$-Y_{18}$	Y_{10}	Y_5	Y_{12}
r_2	$-Y_{14}$	Y_{15}	Y_{21}	$-Y_{26}$	$-Y_{27}$	$-Y_{24}$	$-Y_{14}$	$-Y_{15}$	$-Y_{21}$	Y_{11}	Y_{12}	Y_6

$=$

	1	2
1	$\mathbf{Y}^{11}$	$\mathbf{Y}^{12}$
2	$\mathbf{Y}^{21}$	$\mathbf{Y}^{22}$

TABLE II.

Coefficients of **Y** *for Beam with Arbitrary Cross Section.*

Given: A, B_1, B_2, C, E, L; l, m, n; l_1, m_1, n_1; l_2, m_2, n_2

$Y_1 = \frac{EA}{L} l^2 + \frac{12}{L^3}(B_2 l_1^2 + B_1 l_2^2)$	$Y_4 = \frac{C}{L} l^2 + \frac{4}{L}(B_1 l_1^2 + B_2 l_2^2)$
$Y_2 = \frac{EA}{L} m^2 + \frac{12}{L^3}(B_2 m_1^2 + B_1 m_2^2)$	$Y_5 = \frac{C}{L} m^2 + \frac{4}{L}(B_1 m_1^2 + B_2 m_2^2)$
$Y_3 = \frac{EA}{L} n^2 + \frac{12}{L^3}(B_2 n_1^2 + B_1 n_2^2)$	$Y_6 = \frac{C}{L} n^2 + \frac{4}{L}(B_1 n_1^2 + B_2 n_2^2)$
$Y_7 = \frac{EA}{L} lm + \frac{12}{L^3}(B_2 l_1 m_1 + B_1 l_2 m_2)$	$Y_{10} = \frac{C}{L} lm + \frac{4}{L}(B_1 l_1 m_1 + B_2 l_2 m_2)$
$Y_8 = \frac{EA}{L} ln + \frac{12}{L^3}(B_2 l_1 n_1 + B_1 l_2 n_2)$	$Y_{11} = \frac{C}{L} ln + \frac{4}{L}(B_1 l_1 n_1 + B_2 l_2 n_2)$
$Y_9 = \frac{EA}{L} mn + \frac{12}{L^3}(B_2 m_1 n_1 + B_1 m_2 n_2)$	$Y_{12} = \frac{C}{L} mn + \frac{4}{L}(B_1 m_1 n_1 + B_2 m_2 n_2)$
$Y_{13} = \frac{6}{L^2}(B_2 l_1 m_2 - B_1 l_2 m_1)$	$Y_{16} = \frac{6}{L^2}(B_2 m_1 l_2 - B_1 m_2 l_1)$
$Y_{14} = \frac{6}{L^2}(B_2 l_1 n_2 - B_1 l_2 n_1)$	$Y_{17} = \frac{6}{L^2}(B_2 n_1 l_2 - B_1 n_2 l_1)$
$Y_{15} = \frac{6}{L^2}(B_2 m_1 n_2 - B_1 m_2 n_1)$	$Y_{18} = \frac{6}{L^2}(B_2 n_1 m_2 - B_1 n_2 m_1)$
$Y_{19} = \frac{6}{L^2}(B_2 - B_1) l_1 l_2$	
$Y_{20} = \frac{6}{L^2}(B_2 - B_1) m_1 m_2$	
$Y_{21} = \frac{6}{L^2}(B_2 - B_1) n_1 n_2$	
$Y_{22} = Y_4 + L(nY_{16} + mY_{17})$	$Y_{25} = Y_{10} + L(nY_{20} - mY_{18})$
$Y_{23} = Y_5 + L(lY_{18} - nY_{13})$	$Y_{26} = Y_{11} + L(nY_{15} - mY_{21})$
$Y_{24} = Y_6 + L(mY_{14} - lY_{15})$	$Y_{27} = Y_{12} + L(lY_{21} - nY_{14})$

a Cartesian frame. For the mass tensor **M**

$\mathbf{M}' =$

	x	y	z	p	q	r
x	M					
y		M				
z			M			
p				M'_{pp}	M'_{pq}	M'_{pr}
q				M'_{qp}	M'_{qq}	M'_{qr}
r				M'_{rp}	M'_{rq}	M'_{rr}

where $M'^{pq} = M^{pr}lm + M^{qq}l_1m_1 + M^{rr}l_2m_2$ etc. Also

$\lambda_1^{2'} =$

	x_1	y_1	z_1	p_1	q_1	r_1
x_2	$-I$					
y_2		$-I$				
z_2			$-I$			
p_2		$-Ln$	Lm	$-I$		
q_2	Ln		$-Ll$		$-I$	
r_2	$-Lm$	$-Ll$				$-I$

$=$

	x_1	p_1
x_2	$-I$	
p_2	L	$-I$

$$\lambda_2^{1'} = \begin{array}{c|cc} & x_2 & p_2 \\ x^1 & -I & \\ p_1 & -L & -I \end{array} \qquad (26)$$

The equations of the beam (and the transmission line) along the general reference frame is still the same as along the principal axes (Eq. 20), namely

$$\begin{aligned} \mathbf{F}^{1'} &= \mathbf{Y}^{11'}\mathbf{D}_1' + \mathbf{Y}^{12'}\mathbf{D}_2' \\ \mathbf{F}^{2'} &= \mathbf{Y}^{21'}\mathbf{D}_1' + \mathbf{Y}^{22'}\mathbf{D}_2' \end{aligned} \quad \Big| \quad \begin{aligned} \mathbf{I}^{1'} &= \mathbf{Y}^{11'}\mathbf{E}_1' + \mathbf{Y}^{12'}\mathbf{E}_2' \\ \mathbf{I}^{2'} &= \mathbf{Y}^{21'}\mathbf{E}_1' + \mathbf{Y}^{22'}\mathbf{E}_2' \end{aligned} \qquad (27)$$

Similarly all other relations between matrices, such as eq. 21, still remain valid and unchanged along the general reference frames.

PART IV. THE CONSTRAINED BEAM.

The "Third Generalization Postulate."

Once the equations of a beam lying in a general direction have been established, for many purposes the concepts of λ_2^1 and λ_1^2 are not needed and the symbolic equations can be simplified. (They, however, will again be needed in Part VIII when "General Circuit Parameters" are introduced.) It is as yet necessary to subject the ends of a beam to arbitrary constraints and then to interconnect the beam equations analytically.

If the previous two tensor equations (27) of a beam are used, the analysis becomes rather complex even in terms of tensors (matrices). For simplification, new types of entities will be introduced, "compound tensors" (compound matrices or "partitioned" matrices). The latter are tensors (matrices) in which each component itself is a tensor (matrix) instead of an ordinary scalar. It will be postulated that (T. A. N., p. 543):

"All reasonings and equations of a beam, or of a whole structure, in terms of compound tensors (partitioned matrices) must be the same as those of the simplest element of a beam when expressed in terms of scalars."

The equation of a beam acted upon by a tensile force (Eq. 2) is $F = YD$. A similar compound-tensor equation is found by combining the two tensor equations (27) (leaving out the primes) as

$$\mathbf{F} = \mathbf{YD} \qquad F^{\alpha} = Y^{\alpha\beta} D_{\beta}$$
$$\mathbf{I} = \mathbf{YE} \qquad I^{\alpha} = Y^{\alpha\beta} E_{\beta}$$

$$\mathbf{F} = \begin{array}{|c|c|} \hline \overset{1}{F^1} & \overset{2}{F^2} \\ \hline \end{array} \qquad \mathbf{I} = \begin{array}{|c|c|} \hline \overset{1}{I^1} & \overset{2}{I^2} \\ \hline \end{array} \qquad \mathbf{Y} = \begin{array}{c|c|c|} & \overset{1}{} & \overset{2}{} \\ \hline \overset{1}{} & Y^{11} & Y^{12} \\ \hline \overset{2}{} & Y^{21} & Y^{22} \\ \hline \end{array}$$

$$\mathbf{D} = \begin{array}{|c|c|} \hline \overset{1}{D_1} & \overset{2}{D_2} \\ \hline \end{array} \qquad \mathbf{E} = \begin{array}{|c|c|} \hline \overset{1}{E_1} & \overset{2}{E_2} \\ \hline \end{array}$$

The matrix of **Y** now contains twelve rows and columns (Table I) and **F** contains all the twelve forces applied at both ends of the beam. The beam and the six-wire transmission system are thereby considered as a generalized coil.

It is, however, important to remember that the determinant of **Y** is still zero, and the inverse of **Y** cannot be calculated.

Reference Frame Along Constrained Axes.

Hitherto it has been assumed that at both ends of the beam all the six forces may exist. Let it be assumed that at one or at both ends some of the forces do not appear. For instance:

(1) In a universal joint all three couples, F^p, F^q and F^r, are absent.
(2) In a thrust bearing a couple is absent in a general direction.
(3) In a journal bearing a lateral force and a couple both are absent in an arbitrary direction.

In general, either one or two or three of the forces or couples (or any combination of them) may be absent in a particular direction or plane, and at either or both ends of the beam.

Constraints will be introduced in two steps:

(1) At one (or both) ends of the constrained beam, the three reference axes are so assumed that one or two of them are directed along the prohibited directions or planes.
(2) The axes along which no forces act are eliminated.

Introducing for each end a set of nine direction cosines (with respect to the axes of either the unoriented or the oriented beam)

$$\mathbf{A}_1 = \begin{array}{c|c|c|c|} & \overset{x_1'}{} & \overset{y_1'}{} & \overset{z_1'}{} \\ \hline x_1 & l' & m' & n' \\ \hline y_1 & l_1' & m_1' & n_1' \\ \hline z_1 & l_2' & m_2' & n_2' \\ \hline \end{array} \qquad \mathbf{A}_2 = \begin{array}{c|c|c|c|} & \overset{x_2'}{} & \overset{y_2'}{} & \overset{z_2'}{} \\ \hline x_2 & l'' & m'' & n'' \\ \hline y_2 & l_1'' & m_1'' & n_1'' \\ \hline z_2 & l_2'' & m_2'' & n_2'' \\ \hline \end{array}$$

the following matrix of transformation, with twelve rows and columns, is formed

$$\mathbf{A} = \begin{array}{c|cccc} & 1_t' & 1_r' & 2_t' & 2_r' \\ \hline 1_t & \mathbf{A}_1 & & & \\ 1_r & & \mathbf{A}_1 & & \\ 2_t & & & \mathbf{A}_2 & \\ 2_r & & & & \mathbf{A}_2 \end{array} \tag{28}$$

(The subscripts "t" and "r" stand for "translation" and "rotation.") Multiplying the $\mathbf{Y}$ matrix of the unoriented beam (Table I) as $\mathbf{A}_t\mathbf{Y}A$, the admittance coefficients $\mathbf{Y}'$ of the still unconstrained beam are found but expressed along the directions of constraint.

When *all three* forces or all three couples at either end of the beam are eliminated, no new set of axes have to be introduced. For the mass tensor $\mathbf{M}$ this transformation also has to be performed.

Elimination of Constrained Axes.

The elimination of the constrained axes of a beam is exactly identical to eliminating permanently open-circuited junction pairs from an electrical network (T. A. N., p. 244). The steps are as follows:

(1) Rewrite the $\mathbf{Y}$ matrix so that the axes to be eliminated (one, two or more axes) form the last rows and columns.

(2) Divide the $\mathbf{Y}$ matrix along these axes into four parts as

$$\mathbf{Y} = \begin{array}{c|c} \mathbf{Y}_1 & \mathbf{Y}_2 \\ \hline \mathbf{Y}_3 & \mathbf{Y}_4 \end{array} \tag{29}$$

(3) The new $\mathbf{Y}'$ matrix (containing only as many rows and columns as there are force axes) is found by the formula

$$\mathbf{Y}' = \mathbf{Y}_1 - \mathbf{Y}_2(\mathbf{Y}_4)^{-1}\mathbf{Y}_3. \tag{30}$$

This relation follows from a matric generalization of the scalar reasoning

$$\begin{aligned} \mathbf{I}_1 &= \mathbf{Y}_1\mathbf{E}_1 + \mathbf{Y}_2\mathbf{E}_2 \\ \mathbf{O} &= \mathbf{Y}_3\mathbf{E}_1 + \mathbf{Y}_4\mathbf{E}_2 \end{aligned} \quad , \qquad \begin{aligned} I_1 &= Y_1E_1 + Y_2E_2 \\ O &= Y_3E_1 + Y_4E_2 \end{aligned} \tag{31}$$

Eliminating $\mathbf{E}_2$

$$\mathbf{I}_1 = (\mathbf{Y}_1 - \mathbf{Y}_2\mathbf{Y}_4^{-1}\mathbf{Y}_3)\mathbf{E}_1 \quad | \quad I_1 = (Y_1 - Y_2Y_4^{-1} - Y_3)E_1$$

Examples of constrained beams and their equivalent circuits are given in Figures 15 and 16.

If several beams at a junction are similarly constrained, at such ends the same reference frames must be introduced for all the beams and all the masses meeting at the junction.

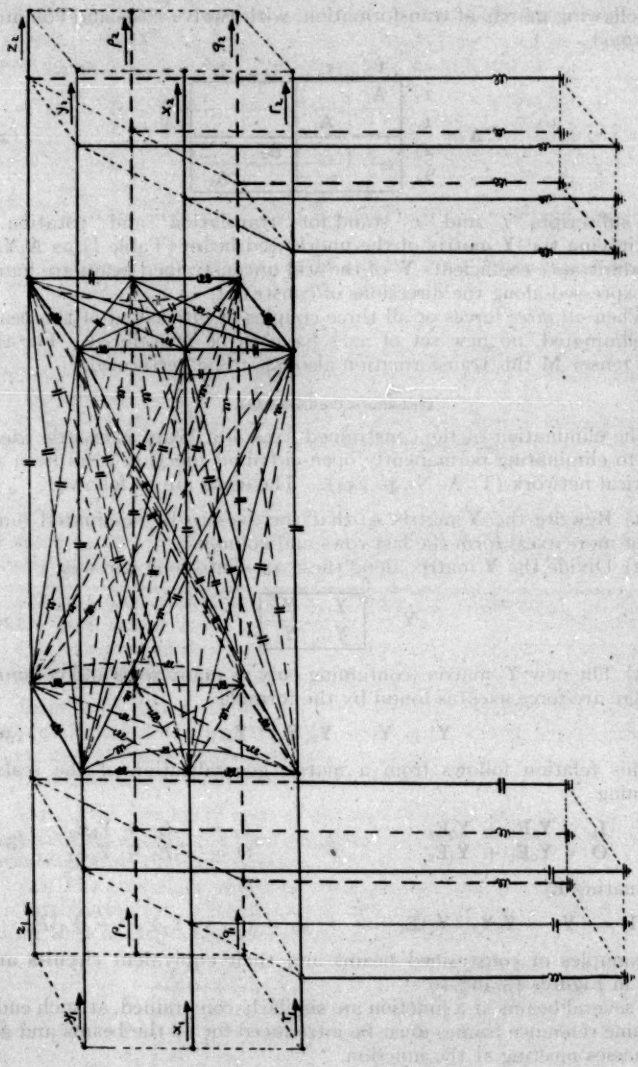


FIG. 8. Equivalent circuit of a beam lying in arbitrary direction (Tables IV and III).

It should be noted that if the original four component matrices $\mathbf{Y}^{12}$, etc., of the beam had been retained, the formula for eliminating the constrained axes would be rather complicated. As a matter of fact, Southwell finds the constants of each type of constrained beam by starting for each case with a new definition of the elastic energy U.

For the mass tensor $\mathbf{M}$ of a mass point lying at the junction, the rows and columns of the prohibited axes are also eliminated by Eq. (30).

PART V. SOLUTION BY EQUIVALENT CIRCUITS.

Equivalent Circuit of a Beam.

With the admittance tensor (matrix) $\mathbf{Y}$ of the unconstrained or constrained beam available, its equivalent circuit, valid for the study of steady state stresses, may be established by the following steps (ref. 4), Fig. 8:

(1) Assume six (or less) terminals at each end point.

(2) The admittance of the coil connecting any two terminals is the corresponding mutual admittance coefficient in $\mathbf{Y}$. (These coils are represented collectively in the compound equivalent circuit, Fig. 4b, by an inductive coil.)

TABLE III.

Coefficients of $\mathbf{Y}$ *for Beam with Circular Cross Section.*

Given: $A, B, C, E, L: l, m, n$			
$F = \frac{EA}{L} - 12\frac{B}{L^3}$		$Y_1 = 12\frac{B}{L^3} + Fl^2$	$Y_4 = 4\frac{B}{L} + Gl^2$
$G = \frac{C-4B}{L}$		$Y_2 = 12\frac{B}{L^3} + Fm^2$	$Y_5 = 4\frac{B}{L} + Gm^2$
$H = \frac{C+2B}{L}$		$Y_3 = 12\frac{B}{L^3} + Fn^2$	$Y_6 = 4\frac{B}{L} + Gn^2$
$Y_7 = Flm$		$Y_{10} = Glm$	$Y_{13} = 6\frac{B}{L^2}n$
$Y_8 = Fln$		$Y_{11} = Gln$	$Y_{14} = -6\frac{B}{L^2}m$
$Y_9 = Fmn$		$Y_{12} = Gmn$	$Y_{15} = 6\frac{B}{L^2}l$
$Y_{16} = -Y_{13}$	$Y_{19} = 0$	$Y_{22} = -2\frac{B}{L}Hl^2$	$Y_{25} = Hlm$
$Y_{17} = -Y_{14}$	$Y_{20} = 0$	$Y_{23} = -2\frac{B}{L}Hm^2$	$Y_{26} = Hln$
$Y_{18} = -Y_{15}$	$Y_{21} = 0$	$Y_{24} = -2\frac{B}{L}Hn^2$	$Y_{27} = Hmn$

(3) The admittance of the coil connecting any terminal to the ground is the *negative* of the sum of the admittances of the corresponding column in **Y**. (Because of this negative sign, the two ground coils in the compound equivalent circuit, Fig. 4b, are represented as capacitors and not as inductors.)

The circuit for an oriented beam (but with no constraints) is given in Fig. 8 (Tables I–IV).

(1) Inductors represent positive numbers.
(2) Capacitors represent negative numbers.
(3) The voltages from line to ground represent displacements of the particular beam ends.
(4) The currents in the lines represent forces at the particular beam ends.

TABLE IV.

Admittances of Coils on Fig. 8 Lying Between Various Junctions

$x_1 \to y_1 = Y_7$	$y_2 \to z_2 = Y_{12}$	$p_2 \to q_2 = Y_{10}$	$z_2 \to p_2 = -Y_{17}$	$y_2 \to z_2 = Y_9$	$x_2 \to y_2 = Y_7$
$z_1 = Y_8$	$y_1 \to z_1 = Y_9$	$r_2 = Y_{11}$	$q_2 = -Y_{18}$	$p_1 = -Y_{16}$	$z_2 = Y_8$
$p_1 = Y_{19}$	$p_1 = Y_{16}$	$z_1 \to p_1 = Y_{17}$	$r_2 = -Y_{21}$	$q_2 = -Y_{20}$	$p_2 = -Y_{19}$
$q_1 = Y_{13}$	$q_1 = Y_{20}$	$q_1 = Y_{18}$	$p_1 \to q_1 = Y_{10}$	$r_2 = -Y_{15}$	$q_2 = -Y_{13}$
$r_1 = Y_{14}$	$r_1 = Y_{15}$	$r_1 = Y_{21}$	$r_1 = Y_{11}$	$q_1 \to r_1 = Y_{12}$	$r_2 = -Y_{14}$
$x_2 = -Y_1$	$x_2 = -Y_7$	$x_2 = -Y_8$	$x_2 = -Y_{19}$	$x_2 = -Y_{13}$	$r_1 \to x_2 = -Y_{14}$
$y_2 = -Y_7$	$y_2 = -Y_2$	$y_2 = -Y_9$	$y_2 = -Y_{16}$	$y_2 = -Y_{20}$	$y_2 = -Y_{15}$
$z_2 = -Y_8$	$z_2 = -Y_9$	$z_2 = -Y_3$	$z_2 = -Y_{17}$	$z_2 = -Y_{18}$	$z_2 = -Y_{21}$
$p_2 = -Y_{19}$	$p_2 = -Y_{16}$	$p_2 = -Y_{17}$	$p_2 = -Y_{22}$	$p_2 = -Y_{25}$	$p_2 = -Y_{28}$
$q_2 = -Y_{13}$	$q_2 = -Y_{20}$	$q_2 = -Y_{18}$	$q_2 = -Y_{25}$	$q_2 = -Y_{23}$	$q_2 = -Y_{27}$
$r_2 = -Y_{14}$	$r_2 = -Y_{15}$	$r_2 = -Y_{21}$	$r_2 = -Y_{28}$	$r_2 = -Y_{27}$	$r_2 = -Y_{24}$

$$\text{ground} \to x_1 = -2(Y_{13}+Y_{14}+Y_{19}) = Y_{gx1} \qquad \text{ground} \to x_2 = -Y_{gx1}$$
$$y_1 = -2(Y_{13}+Y_{16}+Y_{20}) = Y_{gy1} \qquad y_2 = -Y_{gy1}$$
$$z_1 = -2(Y_{17}+Y_{18}+Y_{21}) = Y_{gz1} \qquad z_2 = -Y_{gz1}$$
$$p_1 = (Y_{22}+Y_{25}+Y_{28}) - (Y_4+Y_{10}+Y_{11}) = Y_{gp1} \qquad p_2 = Y_{gp1}$$
$$q_1 = (Y_{23}+Y_{25}+Y_{27}) - (Y_5+Y_{10}+Y_{12}) = Y_{gq1} \qquad q_2 = Y_{gq1}$$
$$r_1 = (Y_{24}+Y_{28}+Y_{27}) - (Y_6+Y_{11}+Y_{12}) = Y_{gr1} \qquad r_2 = Y_{gr1}$$

For the study of small vibrations, in addition to these circuits the circuit of **M** is also needed. It is established in an analogous manner, Fig. 9. All components of **M** are always multiplied by $p^2 = -\omega^2$ where ω is the frequency of the forced or free vibration of the mass point.

It is possible to establish equivalent circuits that correctly represent the beam for *transient* problems such as shocks. These equivalent circuits consist of a twelve-winding transformer for **Y** and a six-winding transformer for **M**. Because of the variable mutual impedances needed, the construction of such transformers is not practical, though theoretically feasible.

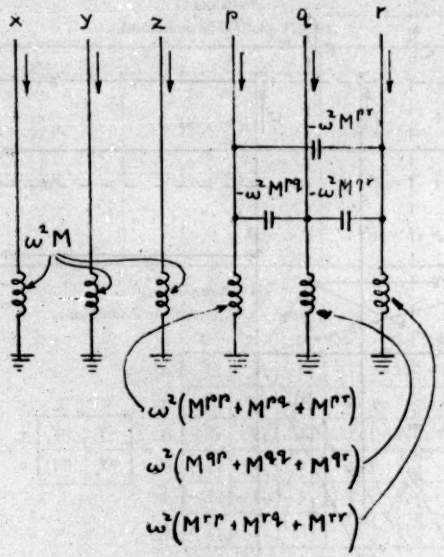


FIG. 9. Equivalent circuit of a rigid body.

TABLE V.

Admittances of Coils on Fig. 10.

$Y_1 = \frac{EA}{L}$	$Y_4 = \frac{C}{L}$
$Y_2 = \frac{12}{L^3}(B_2m_1^2 + B_1m_2^2)$	$Y_5 = \frac{4}{L}(B_1m_1^2 + Bm_2^2)$
$Y_3 = \frac{12}{L^3}(B_2n_1^2 + B_1n_2^2)$	$Y_6 = \frac{4}{L}(B_1n_1^2 + B_2n_2^2)$
$Y_9 = \frac{12}{L^3}(B_2m_1n_1 + B_1m_2n_2)$	$Y_{18} = \frac{6}{L^3}(B_2n_1m_2 - B_1n_2m_1)$
$Y_{12} = \frac{4}{L}(B_1m_1n_1 + B_2m_2n_2)$	$Y_{20} = \frac{6}{L^2}(B_2 - B_1)m_1m_2$
$Y_{15} = \frac{6}{L^2}(B_2m_1n_2 - B_1m_2n_1)$	$Y_{21} = \frac{6}{L^2}(B_1 - B_1)n_1n_2$
$Y_{22} = Y_4$	$Y_{24} = Y_6 - LY_{15}$
$Y_{23} = Y_5 + LY_{18}$	$Y_{27} = Y_{12} + LY_{21}$

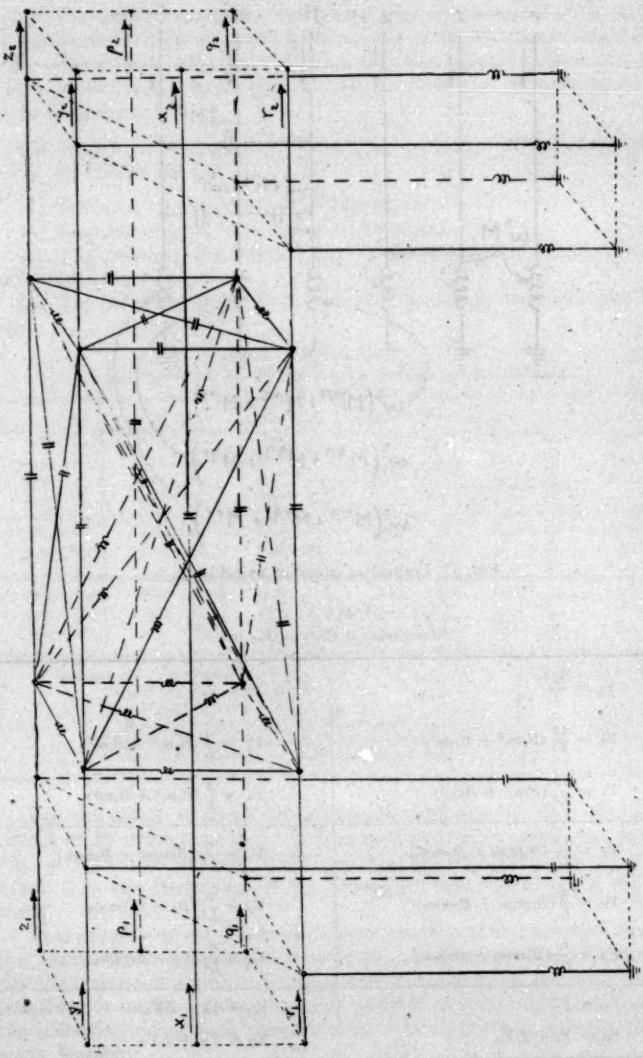


FIG. 10. Three-dimensional beam along *x*-axis, principal axes in *arbitrary* direction (Table V).

TABLE VI.

Admittances of Coils on Fig. 11.

$Y_1 = \frac{EA}{L}$	$Y_4 = \frac{C}{L}$	$Y_{13} = -\frac{6}{L^2} B_1$
$Y_2 = \frac{12}{L^3} B_2$	$Y_5 = \frac{4}{L} B_1$	$Y_{22} = Y_4$
$Y_3 = \frac{12}{L^3} B_1$	$Y_6 = \frac{4}{L} B_2$	$Y_{23} = Y_5 + LY_{13}$
	$Y_{15} = \frac{6}{L^2} B_2$	$Y_{24} = Y_6 - LY_{15}$

TABLE VII.

Admittances of Coils of Plane Beam of Fig. 12.

$$\mathbf{Y} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \mathbf{Y}^{11} & \mathbf{Y}^{12} \\ 2 & \mathbf{Y}^{21} & \mathbf{Y}^{22} \end{array} =$$

	x_1	y_1	r_1	x_2	y_2	r_2
x_1	Y_1	Y_4	$-Y_5$	$-Y_1$	$-Y_4$	$-Y_5$
y_1	Y_4	Y_2	Y_6	$-Y_4$	$-Y_2$	Y_6
r_1	$-Y_5$	Y_6	Y_3	Y_5	$-Y_6$	Y_7
x_2	$-Y_1$	$-Y_4$	Y_5	Y_1	Y_4	Y_5
y_2	$-Y_4$	$-Y_2$	$-Y_6$	Y_4	Y_2	$-Y_6$
r_2	$-Y_5$	Y_6	Y_7	Y_5	$-Y_6$	Y_3

(a) Admittance matrix $\mathbf{Y}$

Given: A, B, E, L; l, m	
$Y_1 = \frac{EA}{L} l^2 + \frac{12B}{L^3} m^2$	$Y_5 = \frac{6B}{L^2} m$
$Y_2 = \frac{EA}{L} m^2 + \frac{12B}{L^3} l^2$	$Y_6 = \frac{6B}{L^2} l$
$Y_3 = \frac{4B}{L}$	$Y_7 = \frac{2B}{L}$
$Y_4 = \left(\frac{EA}{L} - \frac{12B}{L^3}\right) lm$	

(b) Beam along general direction

$Y_1 = \frac{EA}{L}$	$Y_5 = \frac{6B}{L^2}$
$Y_2 = \frac{12B}{L^3}$	$Y_7 = \frac{2B}{L}$
$Y_3 = \frac{4B}{L}$	

(c) Beam along x

$Y_1 = \frac{12B}{L^3}$	$Y_5 = \frac{6B}{L^2}$
$Y_2 = \frac{EA}{L}$	$Y_7 = \frac{2B}{L}$
$Y_3 = \frac{4B}{L}$	

(d) Beam along y

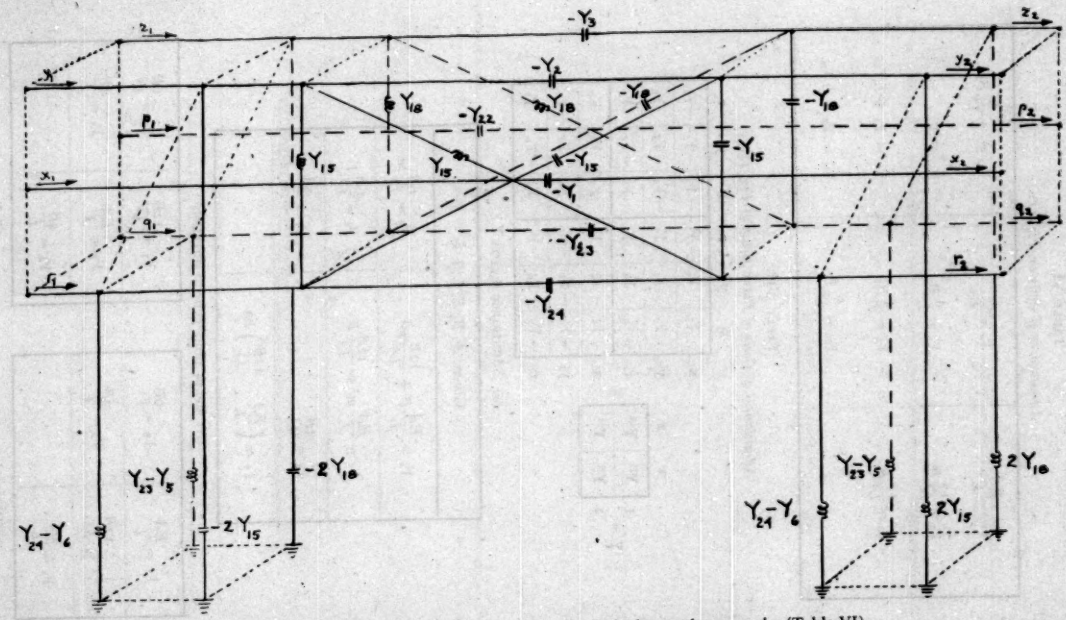
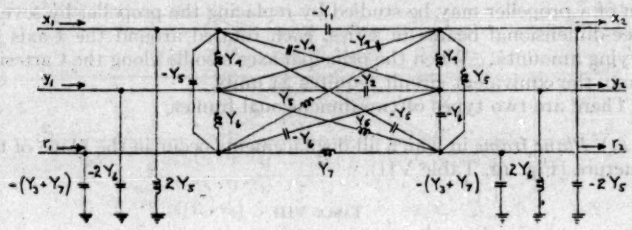
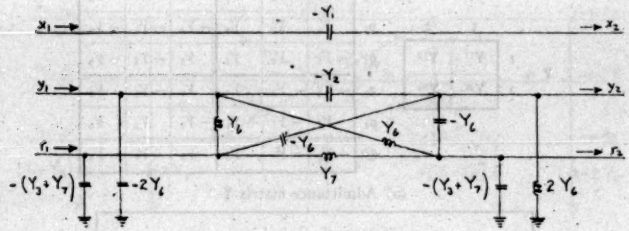


FIG. 11. Three-dimensional beam along x-axis, principal axes along y and z (Table VI).

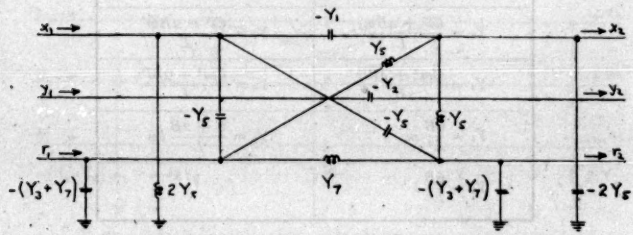
V.2
194



a) Beam along general direction (Table VII6)



b) Beam along "x" (Table VIIc)



c) Beam along "y" (Table VIId)

FIG. 12. Two-dimensional beam in *plane* frame. Both ends rigid.

Special Cases.

The number of coils is greatly reduced when the beam lies along the x, y or z axes with two of its principal axes still in general directions (Fig. 10 and Table V). For instance, the natural frequency of vibra-

tion of a propeller may be studied by replacing the propeller by several three-dimensional beams in series, each rotated around the x-axis by varying amounts. When the principal axes also lie along the Cartesian frame, the equivalent circuit requires 24 units.

There are two types of two-dimensional frames:

(1) *Plane frame* in which all displacements occur in the plane of the structure (Fig. 12, Table VII).

TABLE VIII.

Admittances of Coils of Grid Beam of Fig. 13.

$$\mathbf{Y} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \mathbf{Y}^{11} & \mathbf{Y}^{12} \\ 2 & \mathbf{Y}^{21} & \mathbf{Y}^{22} \end{array} =$$

	z_1	p_1	q_1	z_2	p_2	q_2
z_1	Y_1	Y_4	$-Y_5$	$-Y_1$	Y_4	$-Y_5$
p_1	Y_4	Y_2	Y_6	$-Y_4$	$-Y_7$	$-Y_9$
q_1	$-Y_5$	Y_6	Y_3	Y_5	$-Y_9$	$-Y_3$
z_2	$-Y_1$	$-Y_4$	Y_5	Y_1	$-Y_4$	Y_5
p_2	Y_4	$-Y_7$	$-Y_9$	$-Y_4$	Y_2	Y_6
q_2	$-Y_5$	$-Y_9$	$-Y_3$	Y_5	Y_6	Y_3

(a) Admittance matrix $\mathbf{Y}$

Given: B, C, L; l, m	
$Y_1 = \frac{12B}{L^3}$	$Y_6 = \frac{C-4B}{L} lm$
$Y_2 = \frac{Cl^2+4Bm^2}{L}$	$Y_7 = \frac{Cl^2-2Bm^2}{L}$
$Y_3 = \frac{Cm^2+4Bl^2}{L}$	$Y_8 = \frac{Cm^2-2Bl^2}{L}$
$Y_4 = \frac{6B}{L^2} m$	$Y_9 = \frac{C+2B}{L} lm$
$Y_5 = \frac{6B}{L^2} l$	

(b) Beam along general direction

$Y_1 = \frac{12B}{L^3}$	$Y_5 = \frac{6B}{L^2}$
$Y_2 = \frac{C}{L}$	$Y_7 = \frac{C}{L}$
$Y_3 = \frac{4B}{L}$	$Y_8 = -\frac{2B}{L}$

(c) Beam along x

$Y_1 = \frac{12B}{L^3}$	$Y_4 = \frac{6B}{L^2}$
$Y_2 = \frac{4B}{L}$	$Y_7 = -\frac{2B}{L}$
$Y_3 = \frac{C}{L}$	$Y_8 = \frac{C}{L}$

(d) Beam along y

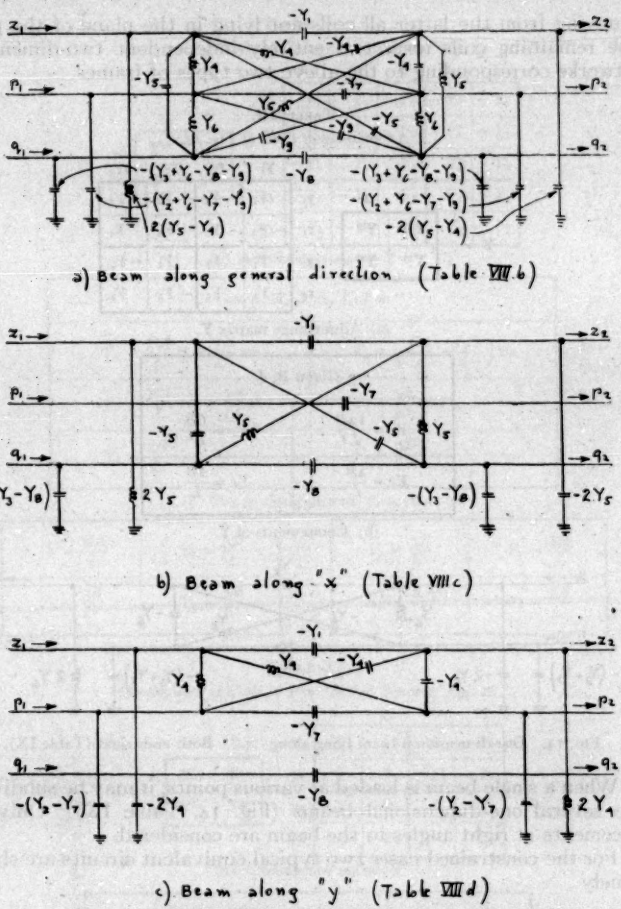


FIG. 13. Two-dimensional beam in *grid*-frame. Both ends rigid.

(2) *Grid frame* in which all displacements are perpendicular to the plane of the structure (Fig. 13, Table VIII).

It should be noted that each of the above two networks may be found from that of the three-dimensional beam (Figs. 8, 10 and 11) by

removing from the latter all coils not lying in the plane of the paper. The remaining coils form two entirely independent two-dimensional networks corresponding to the above two types of frames.

TABLE IX.

Admittances of Coils of Fig. 14.

$$Y = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & Y^{11} & Y^{12} \\ 2 & Y^{21} & Y^{22} \end{array} = \begin{array}{c|cccc} & y_1 & r_1 & y_2 & r_2 \\ \hline y_1 & Y_1 & -Y_3 & -Y_1 & -Y_3 \\ r_1 & -Y_3 & Y_2 & Y_3 & Y_4 \\ y_2 & -Y_1 & Y_3 & Y_1 & -Y_3 \\ r_2 & -Y_3 & Y_4 & -Y_3 & Y_2 \end{array}$$

(a) Admittance matrix **Y**

Given B, L	
$Y_1 = \frac{12B}{L^3}$	$Y_3 = \frac{6B}{L^2}$
$Y_2 = \frac{4B}{L}$	$Y_4 = \frac{2B}{L}$

(b) Components of **Y**

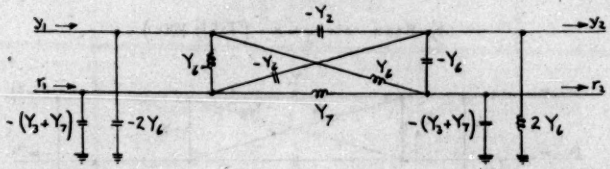


FIG. 14. One-dimensional beam lying along "x." Both ends rigid (Table IX).

When a single beam is loaded at various points, it may be subdivided into several one-dimensional beams (Fig. 14, Table IX). Only displacements at right angles to the beam are considered.

For the constrained cases two typical equivalent circuits are shown, namely

(1) A beam in a plane frame, one end pin jointed, the other end rigid (Fig. 15, Table X).

(2) A beam in a plane frame, both ends pin jointed (Fig. 16, Table XI).

Equivalent Circuit of a Structure.

To represent a whole rigid structure it should be assumed that *the equivalent circuit of each beam lies in space along the whole length of the*

TABLE X.

Admittances of Coils of Constrained Beam of Fig. 15.

$$\mathbf{Y} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \mathbf{Y}^{11} & \mathbf{Y}^{12} \\ 2 & \mathbf{Y}^{21} & \mathbf{Y}^{22} \end{array} =$$

	x_1	y_1	r_1	x_2	y_2
x_1	Y_1	Y_4	$-Y_5$	$-Y_1$	$-Y_4$
y_1	Y_4	Y_2	Y_6	$-Y_4$	$-Y_2$
r_1	$-Y_5$	Y_6	Y_3	Y_5	$-Y_6$
x_2	$-Y_1$	$-Y_4$	Y_5	Y_1	Y_4
y_2	$-Y_4$	$-Y_2$	$-Y_6$	Y_4	Y_2

(a) Admittance matrix **Y**

Given: $A, B, E, L; l, m$	
$Y_1 = \frac{EA}{L} l^2 + \frac{3B}{L^3} m^2$	$Y_4 = \left(\frac{EA}{L} - \frac{3B}{L^3}\right) lm$
$Y_2 = \frac{EA}{L} m^2 + \frac{3B}{L^3} l^2$	$Y_5 = \frac{3B}{L^2} m$
$Y_3 = \frac{3B}{L}$	$Y_6 = \frac{3B}{L^2} l$

(b) Beam along general direction

$Y_1 = \frac{EA}{L}$	$Y_3 = \frac{3B}{L}$
$Y_2 = \frac{3B}{L^3}$	$Y_6 = \frac{3B}{L^2}$

(c) Beam along x

$Y_1 = \frac{3B}{L^3}$	$Y_3 = \frac{3B}{L}$
$Y_2 = \frac{EA}{L}$	$Y_5 = \frac{3B}{L^2}$

(d) Beam along y

TABLE XI.

Admittances of Coils of Pin-Jointed Beam of Fig. 16.

$$\mathbf{Y} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \mathbf{Y}^{11} & \mathbf{Y}^{12} \\ 2 & \mathbf{Y}^{21} & \mathbf{Y}^{22} \end{array} =$$

	x_1	y_1	x_2	y_2
x_1	$-Y_1$	Y_3	$-Y_1$	$-Y_3$
y_1	Y_3	Y_2	$-Y_3$	$-Y_2$
x_2	$-Y_1$	$-Y_3$	Y_1	Y_3
y_2	$-Y_3$	$-Y_2$	Y_3	Y_2

(a) Admittance matrix **Y**

Given: $A, E, L; l, m$		
$Y_1 = \frac{EA}{L} l^2$	$Y_2 = \frac{EA}{L} m^2$	$Y_3 = \frac{EA}{L} lm$

(b) Beam along general direction

$Y_1 = \frac{EA}{L}$

(c) Beam along x

$Y_2 = \frac{EA}{L}$

(d) Beam along y

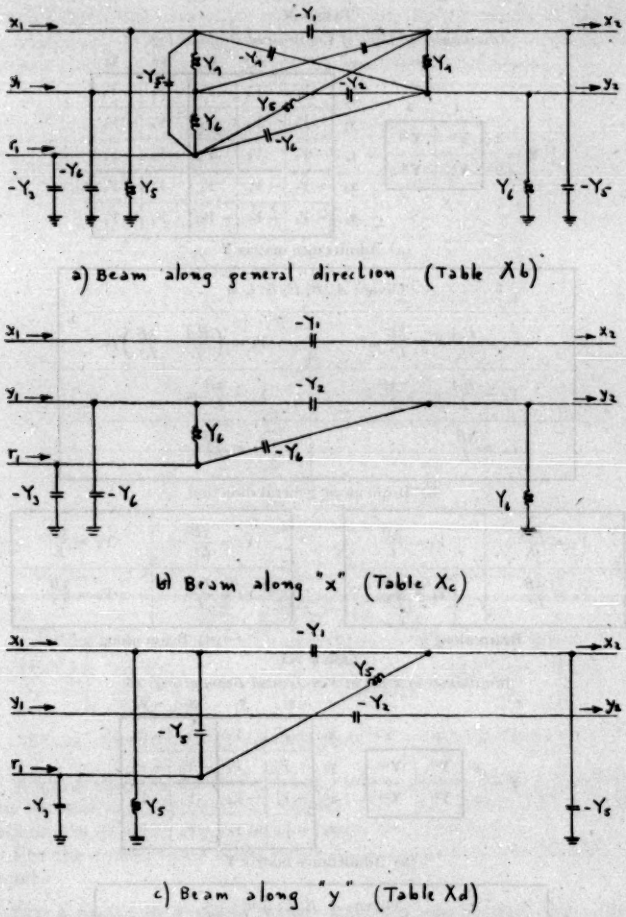
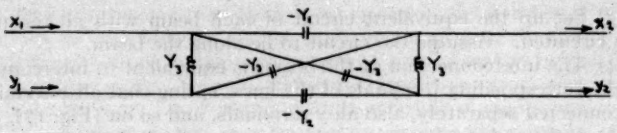


FIG. 15. Two-dimensional beam in *plane* frame. End 1 rigid, end 2 pin-jointed.

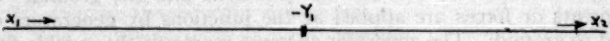
beam. Thereby *the compound electrical network has the same configuration as the elastic network* (Fig. 19). The steps are as follows:

(1) Assume all interconnections of the beams removed.

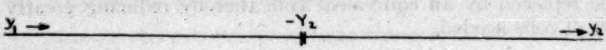
(2) Calculate for each beam in its actual spatial position the admittance coefficients from Tables I–XI.



a) Beam along general direction (Table XI b)

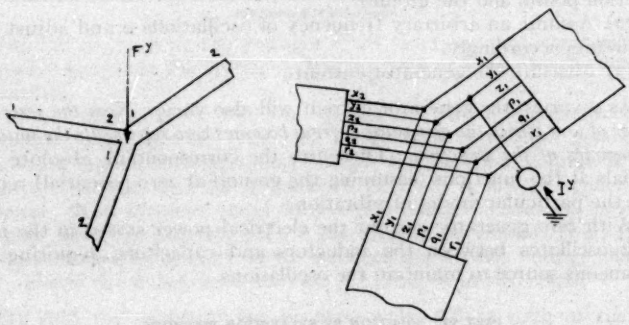


b) Beam along 'x' (Table XI c)



c) Beam along "y" (Table XI d)

FIG. 16. Two-dimensional beam in *plane* frame. Both ends pin-jointed.



a) Interconnection of beams

b) Interconnection of equivalent circuits of beams

FIG. 17.

(3) Set up the equivalent circuit of each beam with all terminals open circuited. Assume the circuit to lie along the beam.

(4) The interconnection of the beams is equivalent to interconnecting the corresponding terminals of the lines, noting that all x terminals are connected separately, also all y terminals, and so on (Fig. 17).

(5) If the end of a beam is fixed against a wall, all the corresponding terminals are shorted to the ground (showing zero voltages and zero displacements).

Mass points existing at the junction points are represented by connecting the **M** circuit between the junctions and the ground. Displacements or forces are applied at the junctions by generators connected to ground. The generator currents represent the outside forces applied and the generator voltages represent the displacements.

When the applied forces have a frequency ω, the **M** network admittances are proportional to ω^2. At zero frequency the **M** circuits are absent.

At the junctions numerous parallel coils belonging to two beams may be replaced by an equivalent coil, thereby reducing greatly the number of coils needed.

Natural Frequencies of Vibration.

The modes and natural frequencies of vibration are determined by the following steps:

(1) Insert a constant-frequency generator between any one of the junction points and the ground.

(2) Assume an arbitrary frequency of oscillations ω and adjust the **M** circuits accordingly.

(3) Measure the generator current.

As ω varies, the generator current will also vary. Now *the series of values of ω at which the generator current becomes zero represents the natural frequencies of the structure*. Of course the corresponding absolute potentials at the junctions (assuming the ground at zero potential) represent the particular modes of vibration.

With zero generator current the electrical power stored in the network oscillates between the inductors and capacitors, requiring no extraneous source to maintain the oscillations.

PART VI. SOLUTION BY RELAXATION METHODS.

The Method of Unbalanced Currents and Voltages.

When the beam structure is replaced by an electrical network, the latter itself may be solved approximately by Southwell's relaxation method. Because of the introduction of compound networks and compound matrices, the relaxation method may be *systematized* with their

aid. It is emphasized that there is no saving in the number of multiplications to be performed by the replacement of a mechanical by an electrical network or by the replacement of a set of scalars by matrices. The systematization, though, eliminates repetitive operations and it speeds up the work by offering a definite schedule of operations.

The relaxation method may be used to improve the accuracy of the results found on the Network Analyzer.

Perhaps the quickest method of solving the network is as follows:

(1) Assume the voltages (displacements) at every junction. (The voltages measured on the Network Analyzer may be used.)

(2) Calculate the currents in each coil.

(3) Add up the currents at every junction.

If the assumed displacements had been correct, the currents would add up to zero at every junction. Because of the wrong guess, an "unbalanced current" i_u exists at every junction whose value indicates the deviation from the correct displacements.

(4) With all unbalanced current i_u available at every junction, a new set of displacements are guessed at.

The guessing and calculation of i_u continues until all i_u are less than any desired value.

The guessing is improved by the introduction of the "unbalanced voltages" e_u at each junction where $e_u = i_u \Sigma y$. That is, i_u is multiplied by the sum of the admittances of all the coils terminating at the junction in question.

The Method of Relaxation.

Instead of changing the voltages at all junctions simultaneously, it is possible to change the voltage at one junction at a time by the following artifice.

Assume that at every junction a generator (with zero impedance) is connected to the ground (Fig. 18a where each compound generator stands for six actual generators). Then the junction voltage is the generator voltage and the unbalanced currents i_u are the generator currents. Now *whenever one of the generator voltages is changed, currents can flow from this generator only to the neighboring grounds* (Fig. 18b). No currents exist elsewhere in the network due to this voltage change ΔE. Hence the unbalanced currents are reduced in two steps:

(1) Add (or subtract) one or more volts $\Delta \mathbf{E}$ to each of the six potentials of a compound junction.

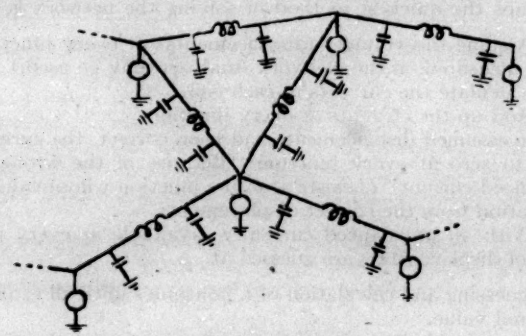
(2) Find the currents at the junction and at all the neighboring junctions.

By knowing the Y-s of each beam, these currents are found for each beam meeting at the junction as follows:

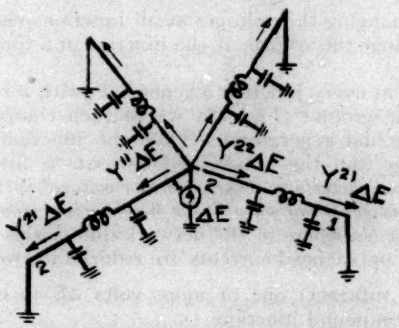
(1) At the generator junction in each beam $\Delta \mathbf{I} = \mathbf{Y}^{11}\Delta \mathbf{E}$ or $\Delta \mathbf{I} = \mathbf{Y}^{22}\Delta \mathbf{E}$ (depending upon whether end 1 or end 2 is at the junction).

(2) At the neighboring junctions in each beam $\Delta \mathbf{I} = \mathbf{Y}^{21}\Delta \mathbf{E}$ or $\Delta \mathbf{I} = \mathbf{Y}^{12}\Delta \mathbf{E}$.

The sum of the old unbalanced currents and these $\Delta \mathbf{I}$-s is the resultant unbalanced current.



a) Hypothetical junction generators



b) Effect of a potential change ΔE

FIG. 18. "Relaxation" method with the use of matrices.

Natural Frequencies of Vibration.

In calculating natural frequencies by the relaxation method, not only the mode of vibration (the junction voltages) are unknown but also the natural frequency of vibration ω corresponding to that mode. However, for every assumed voltage pattern, the corresponding ω may be easily found from the equivalent circuit because of the following properties of the network:

(1) The total electrical power (ΣI^2Z) in the beam networks $\mathbf{Y}$ is equal to the *elastic* energy stored in the whole structure.

(2) the total electrical power in the mass networks $\omega^2\mathbf{M}$ is equal to the *kinetic* energy stored in the structure.

(3) During free vibrations, by Rayleigh's principle

$$\text{Elastic energy} = -\text{ kinetic energy.}$$

Hence ω may be found as follows:

(a) Assume a voltage distribution and unit ω.
(b) Calculate the currents I flowing in each coil.
(c) Calculate the power in each coil I^2Z where $Z = 1/Y$. The sign of Y (plus in an inductor, minus in a capacitor) must be considered.
(d) Calculate separately by summation the elastic and kinetic energies.
(e) Because of their equality

$$\omega^2 = \frac{\text{elastic energy}}{\text{kinetic energy}}. \qquad (32)$$

By knowing the correct ω, the correct unbalanced currents and voltages at every junction may be found.

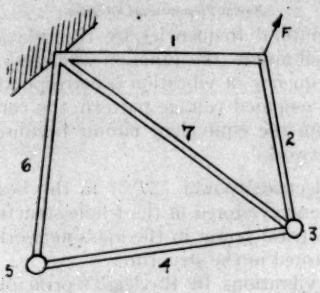
From here on the steps are the same as previously. Either all junction voltages are changed simultaneously, or one at a time. In either case, a new correct ω has to be calculated every now and then.

PART VII. SOLUTION BY ANALYTICAL METHOD.

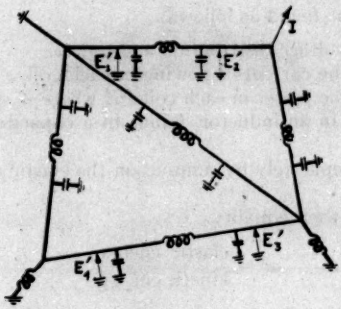
The Primitive System.

The method of establishing the equations of an elastic structure is an *exact* analogue of establishing the equations of an electrical junction network in which each coil itself is a network with twelve or six degrees of freedom. The analysis of such "compound" networks is the same as the analysis of simple networks except that the components of the various tensors $\mathbf{I}$, $\mathbf{Y}$, $\mathbf{E}$ and $\mathbf{A}$ are not scalars but tensors with six degrees of freedom. A brief outline of the method of analysis of the analogous electrical network is illustrated by an example (T. A. N., Chapter XIV).

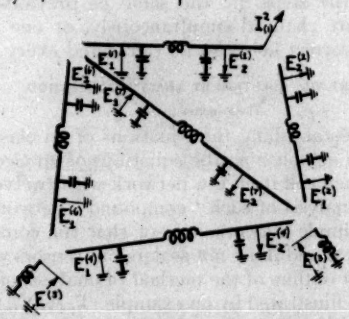
(1) Remove all interconnections between beams and mass points, leaving the "primitive" network. On Fig. 19c there are five elastic beams and two masses.



a) Elastic structure to be analyzed.



b) Equivalent compound-network.



c) The "primitive" network

FIG. 19. The analysis of elastic structures.

(2) Establish the **Y**, **E** and **I** matrices of the primitive network, representing the 72 scalar equations **I** = **YE**. Each elastic beam has two (compound) degrees of freedom, a mass point has one. These matrices of Fig. 19c are

$$\mathbf{Y} = \begin{array}{c|cccccccccccc} & I_1 & I_2 & 2_1 & 2_2 & 3 & 4_1 & 4_2 & 5 & 6_1 & 6_2 & 7_1 & 7_2 \\ \hline I_1 & Y^{11}_{(1)} & Y^{12}_{(1)} & & & & & & & & & & \\ I_2 & Y^{21}_{(1)} & Y^{22}_{(1)} & & & & & & & & & & \\ 2_1 & & & Y^{11}_{(2)} & Y^{12}_{(2)} & & & & & & & & \\ 2_2 & & & Y^{21}_{(2)} & Y^{22}_{(2)} & & & & & & & & \\ 3 & & & & & Y_{(3)} & & & & & & & \\ 4_1 & & & & & & Y^{11}_{(4)} & Y^{12}_{(4)} & & & & & \\ 4_2 & & & & & & Y^{21}_{(4)} & Y^{22}_{(4)} & & & & & \\ 5 & & & & & & & & Y_{(5)} & & & & \\ 6_1 & & & & & & & & & Y^{11}_{(6)} & Y^{12}_{(6)} & & \\ 6_2 & & & & & & & & & Y^{21}_{(6)} & Y^{22}_{(6)} & & \\ 7_1 & & & & & & & & & & & Y^{11}_{(7)} & Y^{12}_{(7)} \\ 7_2 & & & & & & & & & & & Y^{21}_{(7)} & Y^{22}_{(7)} \end{array} \tag{33}$$

$$\mathbf{E} = \begin{array}{c|cccccccccccc} & I_1 & I_2 & 2_1 & 2_2 & 3 & 4_1 & 4_2 & 5 & 6_1 & 6_2 & 7_1 & 7_2 \\ \hline & E_1{}^{(1)} & E_2{}^{(1)} & E_1{}^{(2)} & E_2{}^{(2)} & E^{(3)} & E_1{}^{(4)} & E_2{}^{(4)} & E^{(5)} & E_1{}^{(6)} & E_2{}^{(6)} & E_1{}^{(7)} & E_2{}^{(7)} \end{array}$$

$$\mathbf{I} = \begin{array}{c|cccccccccccc} & I_1 & I_2 & 2_1 & 2_2 & 3 & 4_1 & 4_2 & 5 & 6_1 & 6_2 & 7_1 & 7_2 \\ \hline & & I^2{}_{(1)} & & & & & & & & & & \end{array}$$

The Connection Tensor.

(3) In the resultant network (Fig. 19b) assume as many new voltages as there are junction pairs. (In Fig. 19b there are four junction pairs.)

(4) Set up a relation $\mathbf{E} = \mathbf{AE}'$ between the old and the new voltages. The coefficients give the connection tensor (or matrix of transformation) **A**.

$$\begin{array}{lcl} E_1{}^{(1)} = & E_1' & \\ E_2{}^{(1)} = & -E_2' & \\ E_1{}^{(2)} = & E_3' & \\ E_2{}^{(2)} = & -E_2' & \\ E^{(3)} = & E_3' & \\ E_1{}^{(4)} = & E_4' & \\ E_2{}^{(4)} = & -E_3' & \\ E^{(5)} = & E_4' & \\ E_1{}^{(6)} = & E_4' & \\ E_2{}^{(6)} = & -E_1' & \\ E_1{}^{(7)} = & E_1' & \\ E_2{}^{(7)} = & -E_3' & \end{array} \qquad \mathbf{A} = \begin{array}{c|cccc} & 1' & 2' & 3' & 4' \\ \hline I_1 & I & & & \\ I_2 & & -I & & \\ 2_1 & & & I & \\ 2_2 & & -I & & \\ 3 & & & I & \\ 4_1 & & & & I \\ 4_2 & & & -I & \\ 5 & & & & I \\ 6_1 & & & & I \\ 6_2 & -I & & & \\ 7_1 & I & & & \\ 7_2 & & & -I & \end{array} \tag{34}$$

Each component of **A** is a unit matrix **I** with six rows and columns. (In case of constraints at, say $\mathbf{E}_3'$, all the **I**-s in the corresponding column contain less than six rows and columns.)

The Resultant System.

(4) The $\mathbf{Y}'$ matrix of the resultant network is found by $\mathbf{A}_t \cdot \mathbf{Y} \cdot \mathbf{A} =$

$$\mathbf{Y}' = \begin{array}{c|cccc} & 1' & 2' & 3' & 4' \\ \hline 1' & \mathbf{Y}^{11}_{(1)}+\mathbf{Y}^{22}_{(6)}+\mathbf{Y}^{11}_{(7)} & -\mathbf{Y}^{12}_{(1)} & -\mathbf{Y}^{12}_{(7)} & -\mathbf{Y}^{21}_{(6)} \\ 2' & -\mathbf{Y}^{21}_{(1)} & \mathbf{Y}^{22}_{(1)}+\mathbf{Y}^{22}_{(2)} & -\mathbf{Y}^{21}_{(2)} & \\ 3' & -\mathbf{Y}^{21}_{(7)} & -\mathbf{Y}^{12}_{(2)} & \mathbf{Y}^{11}_{(2)}+\mathbf{Y}_{(3)}+\mathbf{Y}^{33}_{(4)}+\mathbf{Y}^{22}_{(7)} & -\mathbf{Y}^{21}_{(4)} \\ 4' & -\mathbf{Y}^{12}_{(6)} & & -\mathbf{Y}^{12}_{(4)} & \mathbf{Y}^{11}_{(4)}+\mathbf{Y}_{(5)}+\mathbf{Y}^{11}_{(6)} \end{array} \tag{35}$$

(5) The impressed force vector is found by $\mathbf{A}_t \cdot \mathbf{I} =$

$$\mathbf{I}' = \begin{array}{c|cccc} & 1' & 2' & 3' & 4' \\ \hline & & -\mathbf{I}^2_{(1)} & & \end{array} \tag{36}$$

The equations to be solved are $\mathbf{I}' = \mathbf{Y}'\mathbf{E}'$. There are twenty-four scalar equations if the structure is three-dimensional, or twelve equations if two-dimensional.

The equations in matric form may still further be manipulated to reduce the solution of twenty-four equations to the solution of four different sets of six equations. A more detailed analysis and manipulation of such compound junction networks is given in T. A. N., Chapters X and XXII.

PART VIII. GENERAL CIRCUIT PARAMETERS.

Relation Between End Forces.

Of the many possible types of labor saving analysis used by electrical engineers for the study of transmission systems, one will be studied here. The analysis allows the setting up quickly of the equations of two beams *in series;* that is, replacing two beams in series by an equivalent beam. When several three-dimensional beams are in series, as in pipings, successive manipulation reduces the whole structure to one equivalent beam. (In an electrical network such a series connection is possible by simply finding the inverse of **Y** as $\mathbf{Z} = \mathbf{Y}^{-1}$. However, in the mechanical network the inverse of **Y** cannot be calculated as the determinant of the twelve equations $\mathbf{I} = \mathbf{YE}$ is zero. Hence the impedance tensor **Z** with twelve rows and columns does not exist, the elastic frame cannot be analyzed as a "mesh" network and a different method of attack must be introduced for the series connection.)

Let the tensor (matrix) equations (27) of a beam be considered

$$\mathbf{I}^1 = \mathbf{Y}^{11}\mathbf{E}_1 + \mathbf{Y}^{12}\mathbf{E}_2 \qquad \mathbf{I}^2 = \mathbf{Y}^{21}\mathbf{E}_1 + \mathbf{Y}^{22}\mathbf{E}_2$$

$$I^1 = Y^{11}E_1 + Y^{12}E_2 \qquad I^2 = Y^{21}E_1 + Y^{22}E_2$$

To show the value of the "generalization postulates" in complex problems, the following reasoning will be carried on in parallel columns in terms of scalars and tensors (matrices). The purpose of the analysis is to express the end 1 quantities $\mathbf{E}_1$ and $\mathbf{I}^1$ in terms of the end 2 quantities $\mathbf{E}_2$ and $\mathbf{I}^2$.

As a first step let an interesting simplification be introduced. Let $\mathbf{E}_2$ be eliminated from the second equation

$$\mathbf{E}_2 = (\mathbf{Y}^{22})^{-1}(\mathbf{I}^2 - \mathbf{Y}^{21}\mathbf{E}_1) \qquad E_2 = (Y^{22})^{-1}(I^2 - Y^{21}E_1)$$

Substituting into the first equation and rearranging

$$\mathbf{I}^1 = [\mathbf{Y}^{11} - \mathbf{Y}^{12}(\mathbf{Y}^{22})^{-1}\mathbf{Y}^{21}]\mathbf{E}_1 + \mathbf{Y}^{12}(\mathbf{Y}^{22})^{-1}\mathbf{I}^2$$

$$I^1 = [Y^{11} - Y^{12}(Y^{22})^{-1}Y^{21}]E_1 + Y^{12}(Y^{22})^{-1}I^2$$

Because of the existence of the relation $\lambda_2{}^1\lambda_1{}^2 = \mathbf{I}$, it can be proved that the coefficient of $\mathbf{E}_1$ is zero. Since $\mathbf{Y}^{12} = \lambda_2{}^1\mathbf{Y}^{22}$

$$\mathbf{Y}^{11} - \lambda_2{}^1\mathbf{Y}^{21} = \mathbf{Y}^{11} - \mathbf{Y}^{11} = \mathbf{o} \qquad Y^{11} - \lambda_2{}^1Y^{21} = Y^{11} - Y^{11} = 0$$

Hence, *in every beam under any loading*, $\mathbf{I}^1$ *may be expressed in terms of* $\mathbf{I}^2$ *as*

$$\mathbf{I}^1 = \lambda_2{}^1\mathbf{I}^2 \qquad I^1 = \lambda_2{}^1I^2 \tag{37}$$

This product leaves the direct stresses and shear unchanged on both sides of a beam, but changes the bending moments.

General Circuit Parameters.

As the next step let the first equation be rearranged

$$\mathbf{E}_1 = (\mathbf{Y}^{11})^{-1}\mathbf{I}^1 - (\mathbf{Y}^{11})^{-1}\mathbf{Y}^{12}\mathbf{E}_2 \qquad E_1 = (Y^{11})^{-1}I^1 - (Y^{11})^{-1}Y^{12}E_2$$

Since

$$(\mathbf{Y}^{11})^{-1}\mathbf{Y}^{12} = (\mathbf{Y}^{11})^{-1}\mathbf{Y}^{11}\lambda^2{}_1{}_1 \qquad (Y^{11})^{-1}Y^{12} + (Y^{11})^{-1}Y^{11}\lambda_1{}^2$$

the final equations are

$$\mathbf{E}_1 = -\lambda^2{}_1{}_1\mathbf{E}_2 + \mathbf{Z'}_{11}\lambda_2{}^1\mathbf{I}^2 \qquad \mathbf{I}^1 = \lambda_2{}^1\mathbf{I}^2$$

$$E_1 = -\lambda_1{}^2E_2 + Z'_{11}\lambda_2{}^1I^2 \qquad I^1 = \lambda_2{}^1I^2$$

or in terms of "general circuit parameters" (T. A. N., p. 575)

$$\mathbf{E}_1 = \mathbf{A}\mathbf{E}_2 + \mathbf{B}\mathbf{I}^2 \qquad \mathbf{I}^1 = \mathbf{D}\mathbf{I}^2$$

$$E_1 = AE_2 + BI^2 \qquad I^1 = DI^2 \tag{38}$$

where

$$\begin{aligned} \mathbf{A} &= -\lambda^2{}_1, \\ \mathbf{B} &= \mathbf{Z'}_{11}\lambda_2{}^1 \\ \mathbf{D} &= \lambda_2{}^1 \end{aligned} \qquad \begin{aligned} A &= -\lambda_1{}^2 \\ B &= Z'_{11}\lambda_2{}^1 \\ D &= \lambda_2{}^1 \end{aligned} \tag{39}$$

The parameter $\mathbf{C}$ (the coefficient of $\mathbf{E}_2$) is zero. The relation $\mathbf{AD}_t = -\mathbf{I}$ (unit tensor) should be noted. Since $\mathbf{Z'}_{11} = (\mathbf{Y}^{11})^{-1}$ the steps of changing from circuit parameters to admittances and vice versa are evident.

Connecting Two Beams in Series.

Let the circuit parameters of two beams be given

$$\begin{aligned} \mathbf{E}_1{}^{(1)} &= \mathbf{A}^{(1)}\mathbf{E}_2{}^{(1)} + \mathbf{B}^{(1)}\mathbf{I}^2{}_{(1)} \\ \mathbf{I}^1{}_{(1)} &= \qquad\qquad \mathbf{D}^{(1)}\mathbf{I}^2{}_{(1)} \end{aligned} \qquad \begin{aligned} E_1{}^{(1)} &= A^{(1)}E_2{}^{(1)} + B^{(1)}I^2{}_{(1)} \\ I^1{}_{(1)} &= \qquad\qquad D^{(1)}I^2{}_{(1)} \end{aligned}$$

$$\begin{aligned} \mathbf{E}_1{}^{(2)} &= \mathbf{A}^{(2)}\mathbf{E}_2{}^{(2)} + \mathbf{B}^{(2)}\mathbf{I}^2{}_{(2)} \\ \mathbf{I}^1{}_{(2)} &= \qquad\qquad \mathbf{D}^{(2)}\mathbf{I}^2{}_{(2)} \end{aligned} \qquad \begin{aligned} E_1{}^{(2)} &= A^{(2)}E_2{}^{(2)} + B^{(2)}I^2{}_{(2)} \\ I^1{}_{(2)} &= \qquad\qquad D^{(2)}I^2{}_{(2)} \end{aligned} \tag{40}$$

When they are connected in series

$$\begin{aligned} \mathbf{E}_1{}^{(2)} &= -\mathbf{E}_2{}^{(1)} \\ \mathbf{I}^1{}_{(2)} &= \mathbf{I}^2{}_{(1)} \end{aligned} \qquad \begin{aligned} E_1{}^{(2)} &= -E_2{}^{(1)} \\ I^1{}_{(2)} &= I^2{}_{(1)} \end{aligned} \tag{41}$$

Substituting the second set into the first set

$$\begin{aligned} \mathbf{E}_1{}^{(1)} &= \mathbf{A'}\mathbf{E}_2{}^{(2)} + \mathbf{B'}\mathbf{I}^2{}_{(2)} \\ \mathbf{I}^1{}_{(1)} &= \qquad\qquad \mathbf{D'}\mathbf{I}^2{}_{(2)} \end{aligned} \qquad \begin{aligned} E_1{}^{(1)} &= A'E_2{}^{(2)} + B'I^2{}_{(2)} \\ I^1{}_{(1)} &= \qquad\qquad D'I^2{}_{(2)} \end{aligned} \tag{42}$$

where the parameters of the resultant single beam are found by the following matrix manipulations

$$\begin{aligned} \mathbf{A'} &= -\mathbf{A}^{(1)}\mathbf{A}^{(2)} \\ \mathbf{B'} &= \mathbf{B}^{(1)}\mathbf{D}^{(2)} - \mathbf{A}^{(1)}\mathbf{B}^{(2)} \\ \mathbf{D'} &= \mathbf{D}^{(1)}\mathbf{D}^{(2)} \end{aligned} \qquad \begin{aligned} A' &= -A^{(1)}A^{(2)} \\ B' &= B^{(1)}D^{(2)} - A^{(1)}B^{(2)} \\ D' &= D^{(1)}D^{(2)} \end{aligned} \tag{43}$$

These parameters may be changed to admittances by Equations (39) and (21).

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NETWORK ANALYZER SOLUTION OF THE EQUIVALENT CIRCUITS FOR ELASTIC STRUCTURES.

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SYNOPSIS.

The equivalent circuits of elastic structures developed in a companion paper are numerically checked for two known frames. Both structures are also measured on the A-C Network Analyzer for the case of known impressed forces. In each case the measured values checked very satisfactorily with the calculated values, showing that existing analyzers can be used to good advantage to determine the forces and displacements in statically indeterminate structures.

FIRST NUMERICAL CHECK.

In order to check the correctness of the equivalent circuits of elastic structures, two examples were worked out numerically for cases where the answer was already known. The two-dimensional structure of Fig. 1, worked out by Southwell (ref. 1, p. 48), was selected.

The members AB, BC and CD are rigidly connected and have the following constants

$$L = 50; \qquad B = 16{,}250; \qquad EA = 13{,}000.$$

The diagonal member AC is pin-jointed at both ends and has the constants

$$L = 70.7; \qquad EA = 19{,}500.$$

The equivalent circuits of the *horizontal* members are given in Fig. 12b of the companion paper and those of the *vertical* member in Fig. 12c. The values of the admittances are given in Tables VIIc and d, respectively, in terms of the known constants L, B and EA.

For the *diagonal* member (pin-jointed), the equivalent circuit is given in Fig. 16a of the companion paper, and the admittances in Table XIb. For this member lying at 45 degrees, $m = 1/\sqrt{2}$ and $l = -1/\sqrt{2}$.

The resultant network is given in Fig. 2. The network is grounded at the points where the members are fixed in the wall. As the coils connecting grounded points are short circuited upon themselves, they are left out of the diagram.

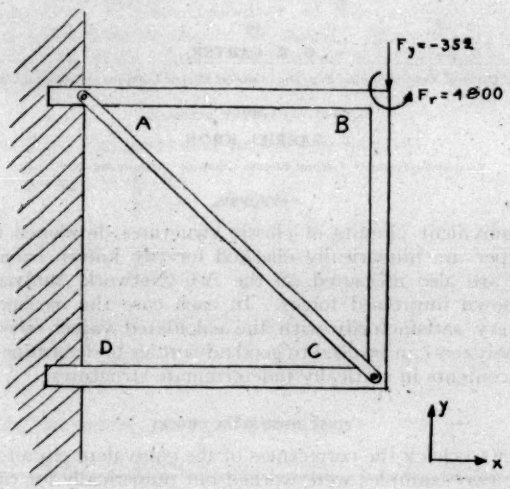

FIG. 1. Given frame.

Southwell impresses at junction *B* a vertical force of 352 units and a bending moment of 4800 units. They are represented on Fig. 3 as impressed currents. He also gives for the linear and angular displacements of the six junctions the figures shown in circles on Fig. 3. They represent the absolute potentials of the junctions, or rather the differences of potentials **E** between the junctions and the ground.

The numerical check consists of calculating all the currents in the network. If the circuit is correct, the sum of the currents at every junction must be either zero or equal to the impressed currents (impressed force) existing at that junction.

The calculated currents are shown in Fig. 3. They do add up to zero or to the impressed currents at every junction within the limit of the small unliquidated currents at the junctions still existing in Southwell's calculation. The currents entering the junctions also give the correct tensions, shears and bending moments existing at each end of each member. The figures of Southwell are shown in parenthesis.

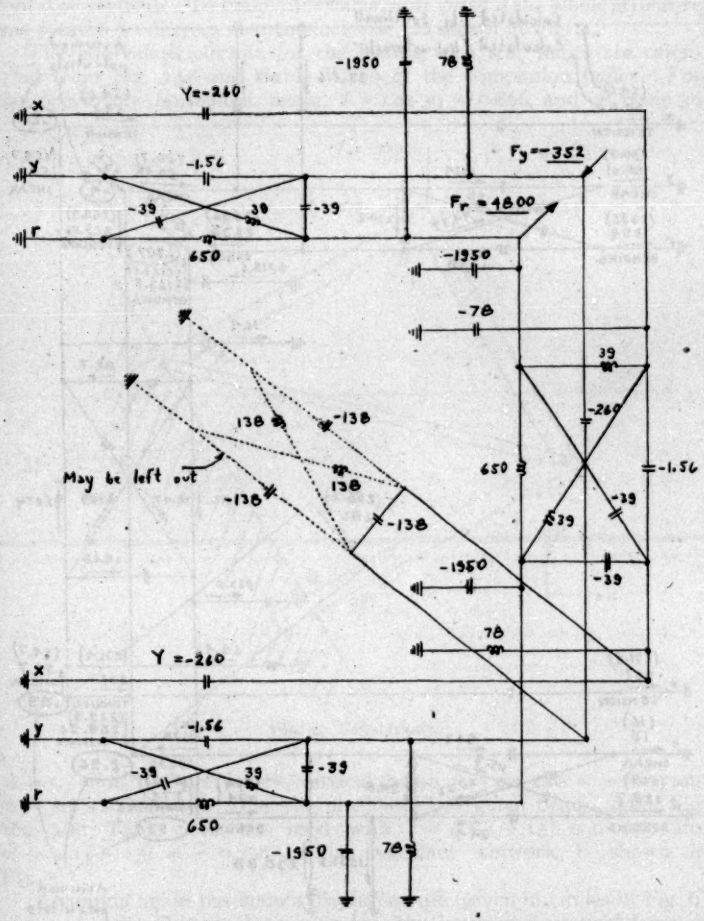

FIG. 2. Equivalent circuit of the frame of Fig. 1.

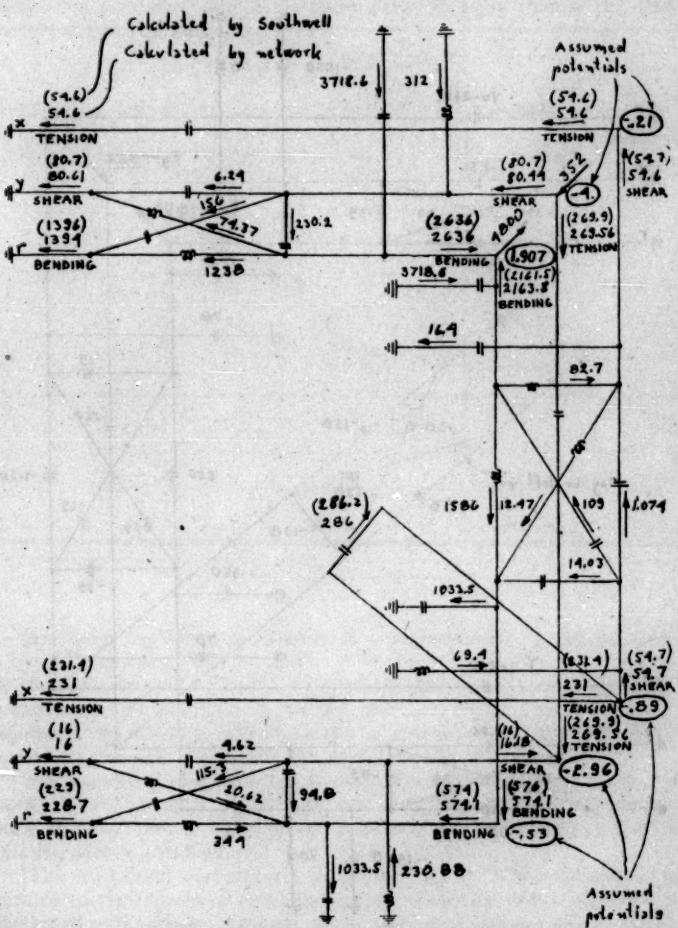

FIG. 3. Calculated currents for assumed potentials.

SECOND NUMERICAL CHECK.

In the previous example the beams with stiff joints were either horizontal or vertical. To make the beams more general the whole structure was rotated 30 degrees counterclockwise, as shown in Fig. 4.

The equivalent circuits for the beams with stiff joints are calculated from Fig. 13a and Table VIIIb of the companion paper. For the previously horizontal beam, $l = \cos 30 = 0.866$ and $m = \sin 30$

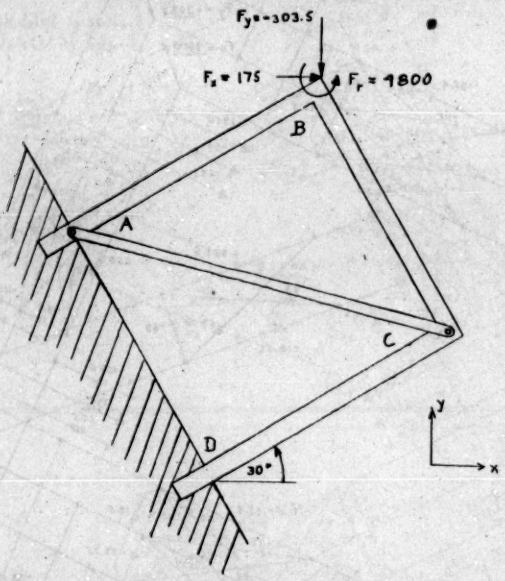

FIG. 4. Given frame.

$= 0.5$. For the previously vertical beam $l = \cos 120 = -0.5$ and $m = \sin 120 = 0.866$. For the pin-jointed diagonal beam, again Fig. 16a and Table XIb are used with $l = \cos(-15) = 0.9659$ and $m = \sin(-15) = -0.2588$. The resultant network is shown in Fig. 5.

Assuming again the known displacements (given in circles in Fig. 6) properly redistributed along x and y, the impressed vertical force of 352 units now appears in two components of 303.5 and 175 units. The calculated currents are given in Fig. 6 and again they add up at the junctions either to zero or to the impressed currents. The forces

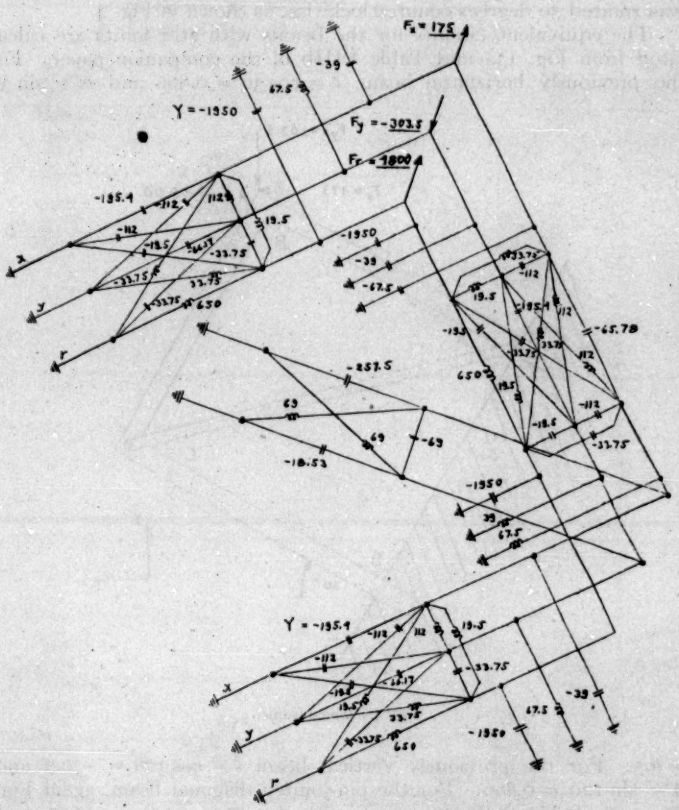

FIG. 5. Equivalent circuit of the frame of Fig. 4.

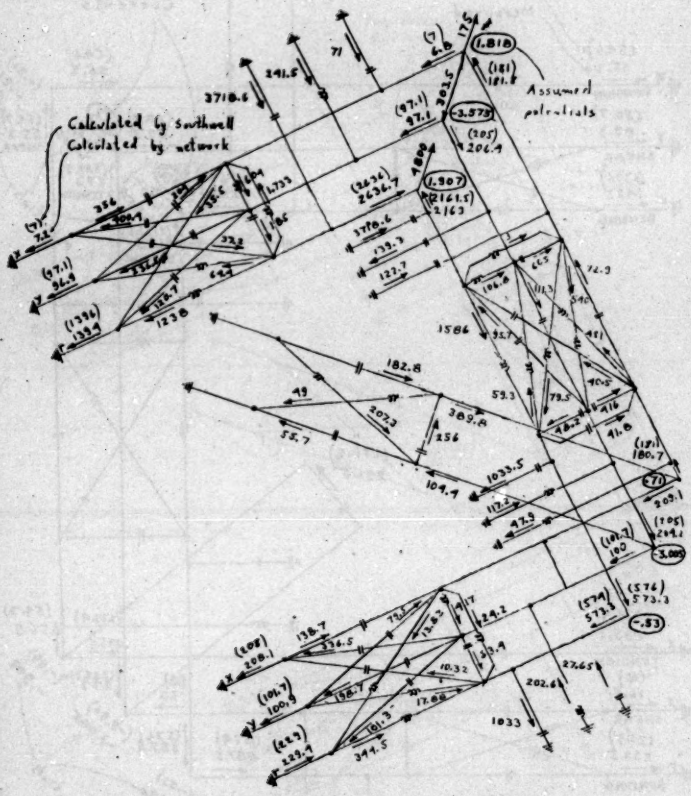

FIG. 6. Calculated currents for assumed potentials.

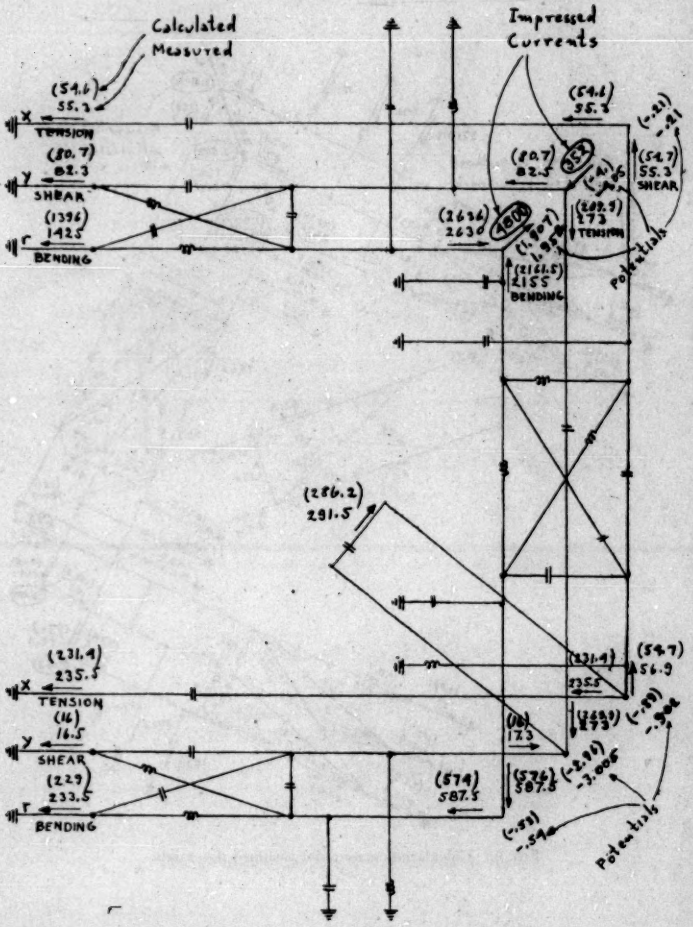

FIG. 7. Network-analyzer solution of the frame of Fig. 1.

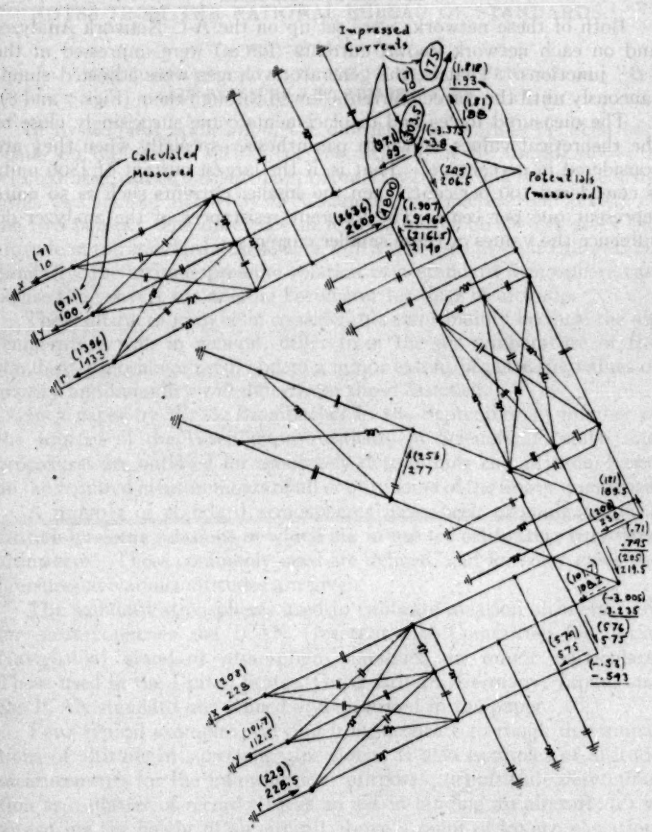

FIG. 8. Network-analyzer solution of the frame of Fig. 4.

at the ends of the beams check the calculation of Southwell (given in parenthesis).

NETWORK ANALYZER TESTS.

Both of these networks were set up on the A-C Network Analyzer and on each network known currents (forces) were impressed at the "*B*" junctions. That is, the generator voltages were adjusted simultaneously until the correct currents flowed through them (Figs. 7 and 8).

The measured forces and displacements come surprisingly close to the theoretical values shown in parenthesis, especially when they are considered in percentage. That is, if the largest current of 4800 units is considered 100 per cent., then the smaller currents such as 50 units represent one per cent. The inherent resistances of the analyzer do influence the values of these smaller currents.

NOTES FROM THE NATIONAL BUREAU OF STANDARDS.*

DETERMINATION OF ALTITUDE BY MEASUREMENT OF AIR PRESSURE AND TEMPERATURE.

In its simplest application, the difference in altitude between two points on the earth's surface, or the altitude of an aircraft above a selected base, can be measured approximately by reading simultaneously either two aneroid altimeters or two aneroid barometers, one at each of the two points. The difference in reading of the altimeters gives the altitude in the standard atmosphere which was used in their calibration. Similarly, the altitude-pressure relation of a standard atmosphere can be used to convert the aneroid barometer readings to altitude.

These altitudes may be in considerable error mainly because the air temperature will, in general, differ from the air temperature of the standard atmosphere used, and to a minor extent because the values of gravity and humidity will differ from those assumed.

In a paper by W. G. Brombacher in the September 15 number of the Journal of the Washington Academy of Sciences, formulas and procedures are outlined for accurately determining the altitude, based on the required measurements of all or only some of the above quantities.

A number of standard atmospheres have been promulgated, the altitude-pressure relations of which are in use for calibrating surveying altimeters. Those commonly used are defined, and for some cases the pressures at various altitudes are given.

The standard atmospheres used to calibrate aviation altimeters are for most countries the ICAN (International Committee for Aerial Navigation) standard atmosphere, modified in minor particulars. Those used in the United States, Great Britain, Germany, Japan, and the ICAN standard are defined and discussed in the paper.

Four typical examples, varying from accurate to rough determinations of altitude in surveying, are given, as also examples of altitude measurements for the following four purposes: (a) altitude determination as a matter of record; (b) as an aid in landing an aircraft; (c) in measuring the height of an aircraft above a point of known elevation, and (d) in aerial photography, where the altimeter reading corresponding to a selected altitude to be maintained above a base is required.

In all the above examples, the altimeter and other instruments are presumed to be free from error. The errors commonly present in altimeters and aneroid barometers—scale errors, hysteresis, drift, zero

* Communicated by the Director.

shift, and temperature errors—are defined, and minimum values for present day instruments are given.

ALCOHOL AS AN ENGINE FUEL.

In the Bureau's studies of substitute fuels for possible use in parts of the world where petroleum is not readily available, engines have been operated on alcohol, both of conventional grade and in mixtures with water containing as little as 35 per cent. alcohol. The results emphasize the fact that the horsepower of an engine is fixed by the amount of air it gets, rather than the amount of fuel. With alcohol (190 proof), a given engine develops more horsepower by several per cent. than with gasoline, despite the fact that the heat available per pound from the alcohol is less than two-thirds as much as from the gasoline. The increased horsepower results from the fact that on vaporizing in the manifold the alcohol cools the air more than does the gasoline, thus permitting a greater weight of air to enter the cylinders.

Although engine operation is possible with the very low-proof alcohols, disadvantages are: (1) the amounts of fuel used are so great that carburetor connections, jets, fuel lines, fuel pumps, and tanks would have to be increased in size; (2) large amounts of water collect in the engine crank case; (3) spark plugs become moist and fail to fire at low throttle. Spark settings must be changed and the fuel consumption, even based on the amount of alcohol alone, is double that for high-test alcohol.

DEFINITION OF AN INCOMBUSTIBLE MATERIAL.

In building code and similar regulations it is necessary to define an incombustible material. A number of definitions are given in American building codes and one based on a specific testing procedure is given in British Standard 476–1932, "British Standard Definitions for Fire-Resistance, Incombustibility, and Non-Inflammability of Building Materials and Structures." The test for incombustibility involves heating in a furnace up to a temperature of 750° C. (1382° F.) in 1½ hours. It was found that the test, thus conducted, would class as incombustible some materials that contained a considerable percentage of combustible constituents and that had been indicated as a fire hazard when used in constructions such as chimneys.

The procedure was modified by heating the furnace to 750° C. before introducing the specimen. This was found consistently to cause flaming and self-sustained glow in materials that have indicated any fire hazard in large-scale tests. For cellulosic materials, the maximum combustible content that can be present if it is to be classed as incombustible by this test, would be only a few per cent. Magnesium is

indicated as combustible, whereas pure aluminum, even in finely-powdered form, is rated as incombustible. However, aluminum paint powder containing about 2 per cent. of oil and less than 1 per cent. of iron oxide, ignited readily.

EXTERIOR PAINT ON TEMPORARY ARMY BUILDINGS.

The exteriors of the temporary buildings at Army cantonments were painted with two coats of paint (with some exceptions) 3½ to 4 years ago. Some of the paint was a titanium-lead-zinc base, some was a lead-zinc base, and a small amount was pure white lead. The present condition of the exterior paint on buildings in various localities was inspected recently so that recommendations could be made concerning the repainting of those buildings that are to be kept in service during the post-war period. The results show that the climate affects the paint differently in different regions. For example, in the arid and sunny regions of the Southwest (for example, Fort Sill, Okla.), paint tends to become very brittle, and on such woods as southern yellow pine, the paint failure takes the form of cracking and flaking. In humid regions of the southern seaboard, paint fails mostly by chalking with little or no flaking. On the other hand, mildew may form on the painted surface in such climates. In the Northeast region, there is an in-between condition where there is usually enough dry weather to cause some flaking of paint on southern yellow pine but not so much as in arid regions. One interesting observation concerned the painting over knots in the wood. The majority of the buildings were built with lumber of No. 2 common grade containing many knots. The usual practice of shellacking the knots or of spotting them with aluminum paint did not prove of value. Therefore, for repainting, no special treatment (other than to scrape off any exudation) will be given the knots.

SPECTROPHOTOMETERS.

Quantitative chemical analysis is often a time-consuming task, and the industrial chemist is always interested in short-cut methods. One of these more rapid methods makes use of the color inherent in a solution, or which is produced in many solutions when mixed or treated with certain other chemicals. This color is caused by absorption of light in a region of the spectrum having a color more or less complimentary to that of the solution itself. If, then, the chemist has available a precise method of measuring this absorption of light, he can determine the amount of absorbing material in the solution with considerable precision.

The spectrophotometer is an instrument that will measure the absorption of light in solutions. Such instruments have been used for

this and other purposes for many years, but only recently has their true importance for such work been realized by the industrial chemist. Proof of the present interest in spectrophotometric methods of quantitative analysis was the presentation (by request) of a paper on spectrophotometers at the symposium on analytical colorimetry and photometry held in connection with the 47th Annual Meeting of the American Society for Testing Materials. In this paper, K. S. Gibson, chief of the Bureau's photometry and colorimetry section, gives descriptions of the visual and photoelectric spectrophotometers at present available to the industrial chemist for analytical use, together with a brief consideration of various factors affecting the accuracy of spectrophotometric work. The paper will appear in the Proceedings of the Society for 1944.

REFLUX REGULATOR AND HEAD FOR LABORATORY DISTILLING COLUMNS

A reflux regulator and head for laboratory distilling columns is described in RP1615 by C. B. Willingham, research associate of the American Petroleum Institute, and Frederick D. Rossini, in the November number of the *Journal of Research*. This regulator is similar to the one described in *J. Research* NBS, **23**, 509 (1939) except that then the limited height of the laboratory made necessary a design that required only about 6 inches between the ceiling and the top of the packing in the rectifying section of the distilling column. This enforced certain compromises in design that were no longer necessary when the laboratory was moved into a new location. The present paper describes apparatus for use in rooms having adequate height. The improvements consist in the better measurement of the temperature of the liquid-vapor equilibrium, in reduction of hold-up, and in increase in mechanical strength.

PERMANENT MAGNETS.

The range of usefulness of permanent magnets has been very greatly extended in recent years by the development of new and superior materials for their construction. However, the mere substitution of a new material without a corresponding modification in design generally leads to disappointing results. For some applications, it may even be better to use the older, less expensive materials. In view of the many requests that the Bureau receives for information on this subject, a publication, Circular C448, has been prepared by Raymond L. Sanford, giving a summary of the data available in the technical literature, and a brief discussion of the design and testing of permanent magnets. Copies are obtainable from the Superintendent of Documents, Government Printing Office, Washington 25, D. C. The price is 10 cents.

METHOD FOR BRINGING ATTACK ON PLATINIFEROUS MATERIALS INTO SOLUTION.

A persistently troublesome problem for the chemist concerned with the analysis of platinum-metal alloys (those that are artificially produced as well as natural minerals) has been to find ways of getting the more refractory of these materials into solution. The best methods hitherto available for this purpose have been laborious and not wholly satisfactory from the analyst's view-point. In a report by Edward Wichers, William G. Schlecht, and Charles L. Gordon, which appears in the November issue of the *Journal of Research* (RP1614), it is shown that even iridium, the most refractory of all the metals of the platinum group, can be brought into solution very satisfactorily if the metal is attacked with a suitable acid mixture at temperatures ranging from 250° to 300° C. Mixtures of hydrochloric acid with nitric acid, perchloric acid, sodium chlorate, or chlorine serve equally well if attention is given to the proper proportioning of the mixtures. At the high temperatures used the acid mixtures develop very high pressures (up to 3500 lb./in.^{2}); consequently, the treatment must be conducted in a sealed glass tube which is enclosed in a steel pressure vessel containing a compressed gas to keep the glass tube from bursting. A known weight of carbon dioxide, placed in the steel vessel in the solid form known as dry ice, conveniently provides the correct compensating pressure.

The rate at which iridium is dissolved by the acid mixture increases rapidly with increasing temperature, and also as the strength of the hydrochloric acid is increased. To obtain the maximum rate of attack for a given temperature and a given strength of the acid, it is necessary to keep the amount of chlorine in the tube, or of any of the substances which serve as sources of chlorine, within rather narrow limits. The rate of attack falls off both when the amount of chlorine is below these limits and when it is above them. This surprising effect was confirmed by an extensive set of experiments, which served also to define the other conditions which should be observed by the analyst in applying the new method of treatment. In the paper are suggested suitable operating conditions for other platinum-metal alloys as well as for the alloys of iridium.

New Principle Assures Efficient Heat.—(*Coal Age*, Vol. 49, No. 9.) A new development in economical, completely automatic, highly compact anthracite heating was demonstrated to home-heating manufacturers at the Anthracite Industries Laboratories at Primos, Pa. recently. Utilization of the new principle will allow production of an automatic unit, suitable for heating the average size home, that can be enclosed in a space less than $2 \times 2 \times 3$ feet.

Dr. Raymond C. Johnson, vice president in charge of research for Anthracite Industries, stated that the principle is founded upon a basic characteristic of anthracite combustion which has been confirmed by research in the laboratory. Anthracite, unlike most other fuels, under proper conditions can be made to burn to complete and perfect combustion within its own area without the necessity for secondary air or secondary combustion spaces.

Heretofore practically all heaters were designed with large grate areas with extensive combustion space above the fire bed so that the large quantities of carbon monoxide which were formed could be burned above the bed with secondary air. As a result, this equipment contained several times the bulk and weight that is necessary in a furnace especially designed to take advantage of anthracite's particular combustion requirements. The complete absence of complicated flue passes or secondary heat absorbing surfaces in this new method of burning does not lower efficiency, because the heated products of combustion flow through the cold incoming coal to stimulate ignition and to use the coal as an enconomizer, replacing conventional secondary heating surfaces.

Equally effective with either hot water, steam or warm air systems, the mechanism for applying the new principle is simplicity itself. It consists of a hollow tube 6 or 8 inches in diameter and approximately 18 inches long. The anthracite is fed into the tube automatically; the coal burns in the center of the tube and the ash is discharged at the other end. Water or air circulated around the hot part of the tube in a small compact jacket carries the heat throughout the house in the same manner in which it is distributed by present-day heating systems.

An added advantage is the fact that the coal cannot clinker. Despite the higher rate of burning, it is not possible for clinkers to form, because the small size of the fire bed permits the water surrounding the tube to carry off the heat so rapidly that the actual temperature of the burning coal is lower than the point at which clinkers form in an anthracite fire.

In adaptation to fully automatic equipment, the anthracite is fed into the tube by a conveyor leading directly from the coal supply. The ash is discharged into enclosed dust proof containers, which would need to be replaced only two or three times a week, or directly into pits prepared for the purpose outside the basement. The amount of heat desired is regulated automatically by thermostatic control.

R. H. O.

THE FRANKLIN INSTITUTE.

STATED MONTHLY MEETING, NOVEMBER 15, 1944.

The stated monthly meeting of The Franklin Institute was held at 8:15 on Wednesday, November 15th, 1944. The President, Mr. Charles S. Redding, presided.

After the playing of the National Anthem, Mr. Redding introduced Mr. B. Hubert Cooper, Chairman of the Speakers' Bureau of the 1945 United War Chest Campaign, who spoke briefly on the work of the campaign, citing a few instances to illustrate the urgent need for funds. He asked all present to again proffer the cooperation which they had so generously given in the past.

The President then called for a report from the Secretary. Dr. Henry Butler Allen stated that since our last Stated Meeting the following increases in the various categories of Membership should be noted:

Active	61	
Associate	30	
Student	22	
Library Sustaining	4	Companies (representing 22 individuals)
Total	117	
Total Membership as of November, 1944	5,420	

Dr. Allen also called to the attention of the meeting the fact that The Franklin Institute had many interesting items for sale at the book stall which could be used for Christmas gifts for young people.

In addition he stated that tickets for the Christmas Week Lectures on the James Mapes Dodge Foundation were also very acceptable gifts; and continued: that Mr. Charles H. Colvin, Director of Engineering Research, New York University, would be the speaker for the Christmas Week Lectures to be given at 3 o'clock on the afternoons of Wednesday, Thursday, and Friday, December 27, 28 and 29th, on the subject "Flying By Instruments." Dr. Allen stated that Mr. Colvin is one of the world's outstanding authorities on the subject and that he had just returned from an extensive tour of the South Pacific, undertaken at the request of the United States Navy. In his Christmas Week Lectures he will use special demonstration devices which are being lent by the Navy for these lectures.

There being no further business, the President introduced the speaker of the evening, Dr. Ellice McDonald, Director of the Biochemical Research Foundation of The Franklin Institute, who gave his annual report on the work of the Foundation.

Dr. McDonald stated that of necessity it was impossible to disclose the war work which is being done in the laboratory. He concentrated on two very vital experiments developed by himself and his research workers—one, a pneumonia vaccine which might prove very valuable in cases where other medicines have no power; and the other, a substitute for blood plasma which is of the greatest importance in treating burns. Dr. McDonald's talk was illustrated by slides and some few experiments.

At the end of the meeting Mr. Redding called for a discussion, stating that the lecturer would be glad to answer any questions.

The meeting was then dismissed with a rising vote of thanks to Dr. McDonald.

HENRY BUTLER ALLEN,
Secretary.

COMMITTEE ON SCIENCE AND THE ARTS.

(Abstract of Proceedings of Stated Meeting held Wednesday, November 8, 1944.)

HALL OF THE COMMITTEE,
PHILADELPHIA, NOVEMBER 8, 1944.

MR. WILLIAM B. COLEMAN *in the Chair.*

The following report was presented for final action:

No. 3139: Sanforizing Process.

This report recommended the award of an Edward Longstreth Medal to Sanford Lockwood Cluett, of Troy, New York, "In view of the fundamental nature and mechanical ingenuity displayed in the development of the process for the preshrinking of woven fabrics, known as Sanforizing."

JOHN FRAZER,
Secretary to Committee.

LIBRARY NOTES.

The Committee on Library desires to add to the collections any technical works that members would wish to contribute. Contributions will be gratefully acknowledged and placed in the library. Duplicates received will be transferred to other libraries as gifts of the donor.

Photostat Service. Photostat prints of any material in the collections can be supplied on request. Orders received in the morning are filled the same day. The average cost for a print 9 × 14 inches is thirty-five cents.

The library and reading room are open on Mondays, Tuesdays, Fridays and Saturdays from nine o'clock A.M. until five o'clock P.M., Wednesdays and Thursdays from two until ten o'clock P.M.

RECENT ADDITIONS.

BIOGRAPHY.

JAFFE, BERNARD. Men of Science in America. 1944.

CHEMISTRY AND CHEMICAL TECHNOLOGY.

BROOKS, BENJAMIN T. Peace, Plenty and Petroleum. 1944.
Chemical Society. Annual Reports on the Progress of Chemistry for 1943. 1944.
DAVIS, DALE S. Chemical Engineering Nomographs. First Edition. 1944.
LANDIS, WALTER S. Your Servant the Molecule. 1944.
READ, JOHN. A Text-Book of Organic Chemistry. 1944.
WARD, ARTEMAS. The Encyclopedia of Food. 1941.
YOST, DON M., AND HORACE RUSSELL, JR. Systematic Inorganic Chemistry. 1944.

DICTIONARIES.

SELL, LEWIS L. English-Spanish Comprehensive Technical Dictionary. 1944.
TWENEY, C. F., AND L. E. C. HUGHES, Editors. Chambers's Technical Dictionary. Revised Edition with Supplement. 1944.

ELECTRIC ENGINEERING.

REICH, HERBERT J. Theory and Applications of Electron Tubes. Second Edition. 1944.
SLURZBERG, MORRIS, AND WILLIAM OSTERHELD. Electrical Essentials of Radio. 1944.

INVENTIONS.

CARLISLE, NORMAN V., AND FRANK B. LATHAM. Miracles Ahead! 1944.

V. 19

MARINE ENGINEERING.

SEWARD, HERBERT LEE, Editor. Marine Engineering. Volume Two. 1944.

MATHEMATICS.

ANDRES, PAUL G., HUGH J. MISER, AND HAIM REINGOLD. Basic Mathematics for Engineers. 1944.

BATEMAN, HARRY. Partial Differential Equations of Mathematical Physics. 1944.

KENNEY, JOHN F. Mathematics of Statistics. 1939.

WATSON, G. N. A Treatise on the Theory of Bessel Functions. Second Edition. 1944.

METALLURGY.

British Aluminium Company, Ltd. Analysis of Aluminium and Its Alloys. 1943.

TEICHERT, ERNEST J. Ferrous Metallurgy. Second Edition. Volume 3. 1944.

METEOROLOGY.

HAURWITZ, BERNHARD, AND JAMES M. AUSTIN. Climatology. First Edition. 1944.

PHYSICS.

JORDAN, PASCUAL. Physics of the Twentieth Century. 1944.

WOOD, ALEXANDER. The Physics of Music. 1944.

RAILROADS.

American Railway Engineering Association. Proceedings. Volume 45. 1944.

THERMODYNAMICS.

ELLENWOOD, FRANK O., AND CHARLES O. MACKEY. Thermodynamic Charts. Second Edition. 1944.

NOTES FROM THE BIOCHEMICAL RESEARCH FOUNDATION.

A Bonus System in a Research Organization.—ELLICE MCDONALD, Director. Such an innovation as a Bonus system in a group of research workers by its very unique quality is bound to encounter criticism. It is only after a number of years' trial that a report of its functioning is given. After all, this organization was one of the first after the University of Wisconsin to advocate issuance of patents upon research discoveries, the proceeds of the patents to go to the furtherance of research.

This suggestion in regard to patents was fiercely fought but now the most recalcitrant universities have plans to protect their discoveries by patents. Even the American Medical Association has a plan for a patent pool for its members. Actually, patents do two things: they control the purity of the product by preventing irresponsible companies from selling cheaper and impure products and, secondly, they bring further funds for research. The discoverer gains reward in increased opportunity and salary, but there is indeed a question as to whether he should not get a more direct recompense. Certainly, no one who frequently injects Insulin for diabetes could grudge the royalty which protects the purity of that product.

Bonuses for research workers is at first accepted with some reluctance by those imbued with the ardor of science. Before implementing such a scheme, I canvassed quite a number of the workers to find out their point of view. One good and quite advanced worker said, "I work all my time and devote all my efforts at my research and a bonus would not make me work any harder." Actually the bonus was installed and this man received one. A check for five hundred dollars was laid on his desk with the remark, "If you still don't approve of bonuses, you may return this check." It did not come back.

Before executing any bonus plan a number of the bonus systems of large companies were studied. I had early experience with a bonus system for my father was all of his mature life a Chief Factor in the Hudson's Bay Company. This company was chartered in 1670 and for three hundred years had a bonus system available mainly to what we would now call the Heads of Departments. The company was very successful but, in the seventies, it did away with the bonus system and this was one reason, and I believe a great one, for a period of depression in its affairs. It is interesting to note that in the Hudson's Bay Company 22 per cent. of the earnings were set aside for bonuses, while in a big and successful modern company, 21 per cent. of the earnings were set aside. The closeness of these percentages with three hundred years

intervening seemed very interesting but I never have been able to find out upon what premises they were based.

However, a bonus system in a commercial company is quite different from a bonus system in a research organization. In the one, the product is measured in dollars and profits: in the other, the product of research is discovery and who can evaluate that? Research accomplishment cannot be immediately measured as to its usefulness and magnitude; sometimes years are required before its real value is appreciated and its application made general.

So while a bonus will reward accomplishment in dollars, this is also done in increase of salary and other appreciation.

First of all, it is necessary to lay down the fundamental purpose of an incentive system such as bonuses. *It is to offer a financial incentive to a worker or a group of workers to produce work of an acceptable quality over and above the acceptable quality comparable within reasonable limits to that of others.*

In research here lies the difficulty. In commercial affairs, it is possible to set up specified quantities or standards and measure the result against that. However, in the higher brackets of companies, the chief men cannot be measured by those specified standards but their usefulness must be a matter of somebody's opinion and so it is in research where there is no yard stick of dollars, profits or red ink.

In research, the standards are often those of opinion. Research of its very nature has to do with new things. Standards cannot be based upon past production records. Since an incentive plan is designed to offer a financial incentive for a worker to produce research of an acceptable quality over and above the normal or usual, it presupposes that the worker will be fairly and adequately compensated for his effort up to and including the specified amount. Bonuses should not be considered in lieu of salary although it must be confessed that in these troubled times of rapid and often sudden increase in cost of living, this has to a minor degree entered into our consideration. It is difficult often with yearly budgets to adjust compensation to sudden fluctuations in living costs.

Before installing the bonus system in 1936, there had been several years' study of the subject by a number of people. The beginning of the Biochemical Research Foundation in 1927 has given the Director opportunity to study various conditions of research organization and the bonus system was one of them, unorthodox as it may seem to some.

A first consideration was that the system should be as fairly and impartially applied as possible and that it should be constant in this.

The procedure is as follows. There are a number of heads of departments who are asked for the recommendations of their workers. The basis upon which this opinion is to be formed is laid down to them and

they are asked to give a fair and unprejudiced opinion in their analysis and to back this up with a statement of the reasons.

These statements then go to the director who adds his opinions particularly in regard to the heads of departments themselves. The director makes plain his opinions and forwards these in written form to a committee of three who are familiar with the organization but not directly involved in it. This Bonus Committee has then the recommendations and the reasons for the recommendations before them with a statement of previous bonuses and the amount of money available for the purpose. The attempt is to make the whole procedure as impartial as possible.

The bonus awards are divided into two classes—(1) Bonus and (2) "Gifts." The first classification of bonus is granted upon three considerations and may amount to any percentage up to a limit of ten per cent. of salary. The considerations are

1. General good work.
2. Research effort.
3. Maintenance of research.

The conditions are wide in order to include all in the organization. Research with us is so often a matter of a group that to reward the leader would be at the expense of his group. So for that reason the term "maintenance of research" is used to mean the support given by members of the group to the problem.

"Research effort" does not imply discovery or success in research for many men work often a long time and add considerable to the sum of knowledge of a subject without achieving any outstanding success. Yet their work is worthwhile and many would not have run their mile if these had not run their furlong. "General good work" is taken to mean consistent and continuous application and achievement above an accepted standard or "over and above the call of duty."

The category of "gifts" is without any limit except the amount of funds available. It is meant to cover discovery and accomplisment in research, capacity in achievement and outstanding success. There is no limit to the reward and the amount is determined first by the director and then scrutinized by the Bonus Committee. It is not always easy to evaluate the merit of a discovery, but the idea is that it should be rewarded. The total amount of "gifts" naturally varies much from year to year dependent on the merit of accomplishment, but it is kept to give rewards which would not be otherwise covered by the bonus given on percentage of the salaries.

All bonus systems depend upon the spirit in which they are applied. When a bonus system which has proved very successful was installed in a certain company, one old timer said, "It will work all right provided

it is a benign autocracy." With ours, the effort is to make it benign and yet as little autocratic as possible.

It should be emphasized that a bonus is not in lieu of salary but a reward and an *incentive*. In regard to the "gifts," the reward phase no doubt enters into it, but the aim is to keep the incentive complexion.

A bonus system may lean toward the side of generosity or toward the part of penny pinching. With the absence of "specified standards" and depending upon fallibility of human opinion, ours has perhaps leaned toward the generous side, but that is planned at first to make all feel that they have a chance to earn a bonus no matter in what capacity they serve. We have in eight years distributed something more than eight per cent. of the total salaries in bonuses and gifts.

In a research such as ours, it is difficult to make classification as to groups of workers in regard to salary. Each is different in regard to capacity and for this reason the discussion of size of salaries is discouraged to avoid heartburnings. The same is true of bonuses and, with the letter of transmissal, it is stated that any discussion about the bonus will militate against future rewards. This has cut down and prevented bonuses in certain cases.

In all, with eight years' experience of a bonus system in a research organization, the decision is to continue it. The monetary reward is only one phase; the fact that the quality of a worker's effort is recognized and appreciated bulks very large. In all, the grant of bonuses has a very definite influence upon morale. While the last word has not been said in this effort in research organization, the experiment is worth continuing.

BOOK REVIEWS.

ORGANIC CHEMISTRY, by Louis Fieser and Mary Fieser. 1091 pages, tables and illustrations, 16 × 24 cms. Boston, D. C. Heath and Company, 1944. Price $8.00.

This book is a text dealing with the development of the fundamental chemistry of the hydrocarbons, alcohols, acids, and other classes of organic compounds. This is done in orderly, connected sequence designed for use after the elementary chemical technique has been acquired. It starts with the Kekulé theory of structure and proceeds through treatments on saturated hydrocarbons, ethylenic and acetylenic hydrocarbons, petroleum and alcohols. A second division may be classed as that which covers the halogen compounds, carboxylic acids, aldehydes and ketones, amines, stereochemistry and ring formation of compounds other than those of the aromatic series. At this point a chapter on rubber appears which discusses the natural as well as the synthetic product. Sugar chemistry is then taken up, followed by fats and waxes, proteins, and the microbiological processes.

The theory of the nature and substitution reactions of aromatic compounds are now discussed after thorough groundwork of preceding topics. The benzene ring is discussed and aromatic amines, halogens, carboxylic acids, and aldehydes and ketones are all given attention. Considerable space has been allotted to the chemistry of the quinones and this is followed by treatments on naphthalene and polynuclear hydrocarbons. Also much space is given to dyes. A brief mention is made of the synthetic fibers and a larger coverage is that on synthetic plastics and resins. Then there is a chapter which consists of a survey of the structures, properties, and physiological significance of the substances falling in the general category of sterol-like compounds or steroids. Isoprenoid compounds, including terpenes and camphors, are next treated. The last two chapters deal with dietary factors and advances in chemotherapy.

The book has 1,043 pages of text. There is an abridged edition which covers most of the subjects in a briefer form. At the end is a comprehensive subject index. The work is a modern, clear presentation.

R. H. OPPERMANN.

CHEMICAL MACHINERY, by Emil Raymond Riegel. 582 pages, photographs and diagrams, 16 × 23 cms. New York, Reinhold Publishing Corporation, 1944. Price $5.00.

There are three parts to the foundation of chemical industry: chemical science, engineering and economics. None can be divorced entirely from the others, and additional fields of knowledge and study must be drawn upon. Chemical machinery embraces parts of all of these in its design, application and cost. The book at hand, on this topic, is a collection of exact and useful information relating to apparatus and devices used in the chemical industries.

The treatment is primarily descriptive, supplemented with many drawings and photographs. The order in which the devices described are arranged is to follow the sequence of events in the chemical plant. At the very beginning there is a brief chapter dealing with factors in plant location. In operation, the progress is from the stock pile or quarry—in the case of solid raw materials—to crushers, grinders, and screens; and then by conveyors to the reaction vessels. For liquid raw materials, the storage tank is the starting point, after which pumps, and perhaps proportioning devices, are required. The same is true of gases, which must be properly cooled and washed; occasionally, volatile solvents must be recovered from them within the plant. Materials must be mixed, liquids as well as solids; at other stages, solids must be separated from liquids by filtration or otherwise. Solutions of solids must be evaporated, crystallized and the crystals dried; mixtures of liquids, on the other hand, are usually separated by distillation. Auxiliary equipment, vacuum-producing devices and heat exchangers are discussed in two chapters; and reaction vessels, especially kettles, are more

closely examined subsequently. The three concluding chapters on instruments follow gas conditioning, hydraulic equipment and high pressure equipment.

The work has several novel features. For instance, each chapter has in its introduction a classification in tabular form. For each type of device, an example is selected and followed long enough to make its construction and operation clear. For a number of kindred devices serving the same general purpose, a critical appraisal of each, with the special adaption it offers, is presented. Another feature is a discussion on the advisability of classifying "drying of materials" as a unit process rather than as a unit operation.

The book represents a well-rounded coverage of the subject. It is up-to-date, clear and well arranged. It is valuable to the industry as a reference to the many types of machinery.

R. H. OPPERMANN.

THERMODYNAMIC CHARTS, by Frank O. Ellenwood and Charles O. Mackey. (Second Edition.) 46 pages, 28 × 21 cms. New York, John Wiley & Sons, Inc., 1944. Price $2.75.

These charts arranged in book form have been made more useful by modifications and additions in this second edition of the book. They include steam charts, water charts which include the pressure scale for the saturated liquid, an entirely new form of chart for steam and water mixtures in which the coördinates are quality as abscissa with saturation pressure and temperature as ordinates, ammonia charts, Freon-12 charts and psychrometric charts. Examples are given of the use of the charts and there are tables of corrections of barometric readings, jet velocity from an ideal nozzle for each B.T.U. and for each tenth of a B.T.U., squares of numbers and common logarithms.

The tables and charts are tools for the solution of many problems. They are clearly presented here and in a form easy to use.

R. H. OPPERMANN.

CHEMICAL ENGINEERING CATALOG 1944–45, Twenty-Ninth Annual Edition. 1580 pages, illustrations, 21 × 28 cms. New York, Reinhold Publishing Corporation, 1944.

The Chemical Engineering Catalog now published in the 29th annual edition will be of much use to those who have need of a concise reference work in the industrial chemical field. The Catalog contains a vast number of manufacturers' catalogs of supplies, equipment and chemicals, also a list of important technical and scientific books and a list of trade names, all very complete and well indexed.

This present edition is enlarged, as many more manufacturers are included than in the previous editions.

P.

LIGHT, VISION AND SEEING, by Matthew Luckiesh. 323 pages, photographs and illustrations, 15 × 22 cms. New York, D. Van Nostrand Company, Inc., 1944. Price $4.50.

The author of this book is well known for his many achievements in the field of illumination. His two recent books, which were reviewed in these pages, have dealt with results of researches, revealing something of the effort to ascertain how light and other controllable factors can best serve human beings. The book at hand contains a simplified presentation of the relationships of light, brightness, vision, lighting, and seeing. There are sixteen chapters, all quite short, into which the subject is divided. The reader is first introduced to facts showing that modern life results in an eyestrain that must be mitigated. There is an appalling prevalence of eye defects which will not continue to be accepted. There is need for seeing specialists and there is a clear field and a great opportunity.

A historical review from the candle to modern lighting in an endeavor to duplicate the sun provides a background for the more detailed discussion on the factors and conditions which make things visible and the effect of brightness, size, brightness contrast. The time factor as an element of seeing was present in all the fundamental researches and its importance is well brought out, as well as a good description of test-charts and their limitations. After this other factors are shown to be serious in seeing conditions, and all of them must be considered when

measuring visibility. The author at this point discusses what he terms the "ease of seeing" which is the criterion and guide in establishing see-levels and in appraising seeing conditions. This involves brightness levels, contrast ratios and the integrated effect of multiple glare sources as well as color. The last chapter is on efficiency, safety and easy seeing revealing much of value and interest.

The book is a combination of fundamental facts with practical discussion prepared for use by those interested in applying better seeing conditions.

R. H. OPPERMANN.

DICTIONARY OF ORGANIC COMPOUNDS, Revised and edited by I. M. Heilbron. 1,072 pages, 18 × 24 cms. New York, Oxford University Press, 1943. Price $30.00. Supplements to Volumes II and III, Price $1.00 each.

Volume I of this well known work is entirely rewritten incorporating the many things which were omitted in the first edition, and later data. The compounds listed are in strict alphabetical order. In the case of those compounds which are named in a variety of ways, the name in which preference is usually given is the name used here for the principal reference. A generous number of cross references is used, and the full constitutional formula is given for every compound whose constitution has been ascertained. Functional derivatives, as distinct from substitution compounds, are given under the parent compound except where they are sufficiently important to justify separate inclusion as principal references. Literature references are given relating primarily to the best method of preparation, but in some cases publications are cited which refer to earlier methods of preparation or which contain useful summaries or bibliographies.

Covering Abietic Acid to Dypnone this 1,072 page volume is a comprehensive and valuable work.

R. H. OPPERMANN.

COLLOID CHEMISTRY, Edited and collected by Jerome Alexander. (Volume V.) 1,256 pages, photographs and illustrations, 16 × 24 cms. Reinhold Publishing Corporation, 1944. Price $20.00.

Volume V of this series of books follows the lines of the earlier volumes and is the forerunner of still another volume. It contains papers dealing with Theory and Methods and Biology and Medicine by a rather complete and distinguished group of contributors, who represent a variety of national origins. The papers cover a broad field within the limitations of the topic heading which, as the editor points out, serves to advance the usefulness of learning. "The all too commonly held notion that a chemical analysis or isolation of an effective chemical substance as an end in itself vanishes when we consider the highly complicated phenomena of biology and of industry."

In the first part—Theory and Methods—there are twenty-five papers. Under Theory there can be cited Rheological Properties of Simple and Colloidal Systems by H. Eyring and R. E. Powell, Colloid Chemical Aspects of Photographic Development by S. E. Sheppard, and Sonic and Ultrasonic Waves in Colloid Chemistry by Karl Sollner. With regard to Methods, there are Methods for the Analysis of Complex Molecular Structures with the Aid of X-Rays by M. L. Huggins, High Speed Centrifugation by E. G. Pickels, and Microradiography of Colloid Materials by G. L. Clark. The second part, dealing with Biology and Medicine, contains thirty-four papers, all authoritative and of interest. The editor has contributed a paper on Catalysis as a Biological Factor and there are others that treat on plant cell membranes, the colloid chemistry of purified viruses, the modern outlook on blood coagulation, the causes and nature of cancer, the action of war gases, and psychiatry. An appendix contains a table of nuclear properties by R. D. Evans. There are author and subject indexes.

The wide variety of topics and the different modes of approach to the same or interrelated subjects enable a broad view to be had which often furnishes an invaluable background for development.

R. H. OPPERMANN.

INFRARED SPECTROSCOPY, by R. Bowling Barnes, Robert C. Gore, Urner Liddel, and Van Zandt Williams. 236 pages, drawings and tables, 16 × 24 cms. New York, Reinhold Publishing Corporation, 1944. Price $2.25.

The fundamental work in proving conclusively that infrared spectroscopy pays its own way in industry has been done. It is rapidly becoming a practical necessity to any research or production organization handling organic chemicals. One of the first industrial infrared spectrometers was constructed in the Stanford Research Laboratories of the American Cyanamid Company in 1936. Numerous applications of this instrument have become familiar to a large number of chemists and the potentialities of the tool have widespread interest.

This book describes analytical methods in use in these laboratories since 1937. They can be used in the plant as well as the laboratory, for control analyses may be made from a by-pass in the production line to give a continuous record of the concentration of one component as a function of time. The output of the spectrometers is an electrical current, which may be used to provide automatic control through suitable relays. The book provides working descriptions of the various experimental techniques involved and a brief discussion of the theoretical basis of infrared spectroscopy. One of the fundamental requirements for the successful application of infrared spectroscopy is a knowledge of the absorption spectra of a large number of pure compounds. To meet this need there is included a library of 363 spectra. This is indexed in two ways: (1) an empirical formula index and (2) an alphabetical index. At the end is a bibliography of 2,701 annotated references, which is divided into an author index and a subject index.

The book is a useful tool for those interested in this work.

R. H. OPPERMANN.

USES AND APPLICATIONS OF CHEMICALS AND RELATED MATERIALS, compiled and edited by Thomas C. Gregory. 459 pages, 16 × 23 cms. New York, Reinhold Publishing Corporation, 1944. Price $9.00.

This is Volume II of a two volume set of unusual reference books. It supplements Volume I and its design has been accompanied by much thought to make it useful. In all there are five parts, based of course on the compilation of uses and applications. The first is an index of uses which covers the compilations of both volumes. This is arranged by classification or grouping according to function (i.e., purpose, uses), and there is much cross indexing, adding greatly in the locating of desired material. The second section of the book gives the uses and applications in industry for 2,642 chemicals and related materials. These data are arranged as follows: (1) The name of the chemical or raw material is printed in bold face; (2) Synonyms, Latin and foreign language equivalents are printed in light face roman; (3) The industries in which it is used are printed in italics; (4) Uses and applications are printed in light-face roman under the italicized industries. To this section of the book there are Addenda and a Glossary defining terms and substances.

Part 3 is an index of synonyms and cross references. This is a means to the solution of the many synonymous-term problems which often involve confusion and waste of time. And Part 4 is an index of patents, U. S. and British, cited in the Uses and Applications section. The arrangement is such that the list is valuable in locating and evaluating patents controlled by a competitor, as well as by foreign firms. The addresses of the owners of the patents constitute Part 5 of the book.

This book will have much use among sellers, users, and makers of chemicals and related materials, as well as a great variety of technical workers, advertising agencies, information bureaus, libraries, etc.

R. H. OPPERMANN.

20TH CENTURY ENGINEERING, by C. H. S. Tupholme. 201 pages, photographs and illustrations, 15 × 23 cms. New York, Philosophical Library, 1944.

The layman at home, even the soldier abroad, has little conception of the broadness of engineering developments. While war naturally urges research and development, it prevents

the normal association and communication which does so much to inform of progress. This book has for its purpose a survey of engineering developments as far as they may be revealed. The author is a well known writer on technical subjects in England and United States, and his work in this book is of surprising scope.

Written for the layman, it covers the more spectacular engineering progress of the past few years, both during and immediately before the war. In the first chapter, under the heading of Mechanical Power, the Velox and the Sulzer monotube boilers are explained among others. Then a wide variety of modern engineering work shop processes are covered, followed by a treatment on air conditioning and refrigeration which includes a description of the Packard automobile cooling, heating, and ventilating units. Chapter four is devoted to the main chemical and metallurgical engineering achievements. The uses of the photo-electric tube, the fluorescent light, the infra-red ray, and various controls are explained under the heading of electrical engineering. Traction, marine engineering and aircraft are other topics covered.

The book makes fascinating and instructive reading for the average person.

R. H. OPPERMANN.

PUBLICATIONS RECEIVED.

Ternary Systems, Introduction to the Theory of Three Component Systems, by G. Masing. Translated by B. A. Rogers. 173 pages, drawings, 15 × 23 cms. New York, Reinhold Publishing Corporation, 1944. Price $4.50.

New Architecture and City Planning, Edited by Paul Zucker. 694 pages, photographs, 16 × 24 cms. New York, Philosophical Library, 1944. Price $10.00.

Radio Direction Finders, by Donald S. Bond. 287 pages, illustrations and photographs, 14 × 22 cms. New York, McGraw-Hill Book Company, Inc., 1944.

Prodigal Genius, The Life of Nikola Tesla, by John J. O'Neill. 326 pages, 15 × 22 cms. New York, Ives Washburn, Inc., 1944. Price $3.75.

Electric Circuits and Fields, by Harold Pender and S. Reid Warren, Jr. 534 pages, illustrations, 14 × 22 cms. New York and London, McGraw-Hill Book Company, Inc., 1943.

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